

# A Stability Issue on Controlled ALOHA System with Capture Channel

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신호 포획현상을 가지는 알로하 시스템의 안정성 고찰

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## ABSTRACT

For the traditional ALOHA system without capture, the Markov chain obtained using the number of backlogged users at each slot is shown to be non-ergodic. So the infinite population ALOHA with fixed retransmission probabilities is unstable for any choice of the arrival rates and retransmission probabilities. The capture ALOHA system is also shown to be unstable for any arrival rate unless it has perfect capture. In this paper, we study a stabilization policy for capture ALOHA system that controls the retransmission probabilities and prove the stability of its multidimensional Markovian model by employing a continuous Lyapunov function, and thus identify the stability region. We also study a delay performance through computer simulation to show the stability for any input rate below the maximum achievable channel throughput.

## 要 約

기존의 알로하 시스템은 불안정한 시스템으로 알려져있다. 또한 포획현상을 가지는 알로하 시스템도 불안정하다는 것이 증명되었다. 본 논문에서는 신호 포획현상을 가지는 알로하 시스템에 전송 제어 알고리즘을 적용하고, 새로운 Lyapunov 함수를 도입하여 다차원 마르코프 연쇄 모델의 안정성을 증명하였다. 부가적으로 시스템의 전송효율을 구하였고, 또한 시뮬레이션을 통하여 패킷 전송지연을 분석하였다.

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## I. INTRODUCTION

We study a slotted ALOHA, packet communication system that operates over a capture channel. In a capture channel, differences in the levels of received power and the times of arrival enable the receiver to successfully receive a packet even when two or more have been transmitted simultaneously.

One of the earliest studies of capture is due to Roberts[1]. He investigated a slotted ALOHA system with and without capture for both satellite and ground radio channels. Metzner[2] exploited the capture phenomenon by dividing users into two groups. Namislo[3] considered the same model but assumed that the capture phenomenon arises because of path loss or shadow and Rayleigh fading. Lee[4] proposed priority free multiple power-level random access schemes for a similar model and showed the improved throughput and delay characteristics of the system. Shwartz and Sidi [5] analyzed the effect of various kinds of errors such as erasures, captures and noise on the slotted ALOHA system. Cidon and Sidi[6], studied the capture effect in the context of collision resolution algorithms.

Recently, a number of researchers have studied the applications of ALOHA systems to spread-spectrum multiple access communication networks. Davis and Gronemeyer[7] have investigated combined power and time of arrival capture in a spread spectrum random access system. Pursley [8] studied the performance of frequency hop transmission in a packet communication network and the effects of multiple-access interference in frequency-hop radio networks. Polydolos and Silvester[11] proposed an analytic framework for the study of random access packet-switched spread-spectrum networks under various link topologies and channel conditions. They anticipated that instability would occur since a similar problem exists with narrow band random access systems. In [9], based on a Markovian model, they

identified the conditions for the instability of the infinite user capture ALOHA model when retransmission control is not employed.

In this paper, we stabilize the capture ALOHA system by employing a decentralized multiplicative control algorithm and determine, after proving its stability, the stability region as a function of the capture parameter.

In section II we obtain a Markov model that describes the fluctuation of the backlog. In section III we comment on the stability issue. Then we formulate the retransmission control policy and compute the throughput as a function of the capture parameter. In section IV, we prove via employing a continuous Lyapunov function that the retransmission control policy leads to a stable system, i.e. the average backlog is finite. In section V computer simulation results are provided, that confirms the stable operation of the system. Finally we summarize our results in section VI.

## II. MARKOVIAN MODEL

### 1. User Model

We assume there are an infinite number of independent and identical users in the system. Each user can have at most one packet waiting to be transmitted at any time: there are no queues associated with the users. The cumulative incoming traffic is Poisson distributed with a mean rate of  $\lambda$  packets per slot.

### 2. Channel Model

The channel is assumed to be noiseless. The time axis is divided into constant intervals called slots. The messages are transmitted in packets whose length are equal to or less than the length of a slot. The channel is such that a receiver can, with some positive probability, decode one of several simultaneous signals on the channel. This broad definition includes capture because of power variations (random or fixed) and because of different times of arrival in spread-spectrum systems. Propagation delays are negligible and

ignored. All users in the system are informed of the outcome of the slots  $\{0, 1, e\}$  by the feedback signal, immediately at the end of a time slot. Additionally, it is assumed that the identity of a captured packet is known to all the users. One possibility is to have the central receiver broadcast the identity of all successfully received packets. Such an assumption has been employed by [6].

### 3. Channel Access Protocol

The Channel access protocol is executed by each user schedule the transmissions of newly arrived packets. We consider IFT (immediate first-transmission) protocol under which new packets that arrive during slot  $t-1$  are first transmitted in slot  $t$  with probability one. Slot  $t$  is the time segment  $(t, t+1]$ .

During each slot, each user with a backlogged packet must decide whether to transmit the packet in that slot. Let  $Z_t$  be the feedback random variable during slot  $(t, t+1]$ . The user is informed via the common feedback at the end of slot  $t$  (without error) that either the empty ( $Z_t=0$ ), the capture slot ( $Z_t=1$ ), or the collision slot ( $Z_t=e$ ). In addition at the end of slot  $t$  each user learns whose packet is captured. Each of two or more users that transmits a packet and does not capture the channel joins the backlog and awaits retransmission at a later time.

We define  $C_n$ , as the probability that one packet is successfully received from among  $n$  simultaneous packets on the channel. Thus  $C_n$ , depends on the number of users transmitting simultaneously.  $C_0=0$  and  $C_1=1$  by definition. Perfect capture occurs if  $C_n=1$  for  $n \geq 2$ .

### 4. Markovian Model

Let  $N_t$  be the number of backlogged packets at time  $t$  and  $P_{ij}=P[N_{t+1}=j|N_t=i]$  be the backlog transition probabilities. We can develop the channel transition probabilities that characterize the dynamics of the channel backlog [9].

Let  $S(n, f)$  be the probability of a successful

packet in a time slot given that  $(N_t=n, f_t=f)$ , that is,

$$S(n, f) = P[Z_t=1|N_t=n, f_t=f]. \tag{2.1}$$

Then we may show that

$$S(n, f) = \sum_{j=0}^i \sum_{m=0}^n \frac{\lambda^j e^{-\lambda}}{j!} \binom{n}{m} f^m (1-f)^{n-m} C_{m+j}. \tag{2.2}$$

Note that by the Poisson theorem, if  $f=f(n)$  depends on  $n$  such that

$$\lim_{n \rightarrow \infty} n f = \mu > 0$$

then

$$\lim_{n \rightarrow \infty} \binom{n}{m} f^m (1-f)^{n-m} = \frac{\mu^m e^{-\mu}}{m!} \text{ for all } m < \infty.$$

This Poisson approximation to binomial distribution is only possible if  $f$  is controlled based on the variations of  $n$ . With this assumption,  $S(n, f)$  of Eq.(2.2) can be approximated by

$$\begin{aligned} S(\mu) &= \sum_{j=0}^i \sum_{m=0}^n \frac{\lambda^j \mu^{-m}}{j! m!} C_{j+m} \\ &= \sum_{k=1}^i \frac{(\lambda + \mu)^{-m}}{k!} e^{-(\lambda + \mu)} C_k. \end{aligned} \tag{2.3}$$

Note that  $S(\mu)$  may appear to follow from the standard Poisson first introduced by Abramson [10] but this is not valid at all unless  $f$  is controlled. Many authors [2], [4], [8], [10], [11], [12] have used this formula in their analysis without fully justifying their approach.

In fact it is shown [9] that for both protocols the chain  $(N_t)_{t \geq 0}$ , is not ergodic unless  $\lim_{n \rightarrow \infty} C_n > 0$ . Accordingly, neither  $S(n, f)$  nor  $S(\mu)$  is an achievable channel throughput and so the throughput in steady-state is zero.

### III. STABILIZATION OF ALOHA SYSTEM WITH CAPTURE

For the traditional ALOHA system without capture, Kaplan [13] proved that the Markov chain obtained, using the number of backlogged users at each slot as the state, is non-ergodic. So the infinite population ALOHA with fixed retransmission probabilities is unstable for any choice of the arrival rate and retransmission probability [14]. In [9], we reported that the capture ALOHA system is also unstable unless  $\lim_{n \rightarrow \infty} C_n > 0$ . The uncontrolled ALOHA system with capture probabilities  $C_n = Q^n$  for some  $0 \leq Q < 1$  is unstable for any arrival rate of  $\lambda$ . Yet, it is not obvious how retransmission control will change the capacity region. Our goal has been to identify the stability region, when a decentralized retransmission control policy is employed.

In this section, we will study a policy that controls the retransmission probabilities and thereby maintains the retransmitted traffic intensity at an optimal level. Our control policy is similar to the one proposed by Hajek and Van Loon [15] and differs from the ones proposed by Rivest [16], Tsitsiklis [17]. We shall assume that  $C_n = Q^n$ . This choice is consistent with [7], [8], and [11] and is also analytically tractable.

#### 1. Retransmission Control Algorithm

Recall that  $Z_t$  denotes the channel feedback information at time  $t+1$ , and that  $f_t$  is the retransmission probability for the backlogged users at time  $t$ . Thus  $f_t$  must be a function of the channel output history  $(Z(s) : s < t)$  for each,  $t$ , and  $0 \leq f_t \leq 1$ . We consider the retransmission control algorithm first proposed in [15] for the standard ALOHA model:

$$f_{t+1} = \{\alpha^\gamma(Z_t) f_t \wedge \beta\} \tag{3.1}$$

for some positive constants  $\gamma, \beta, a(0), a(1)$  and  $a$

( $e$ ). Note that the process  $(N_t, f_t)$  is a two-dimensional Markov process with state space  $Z_+ \times [0, 1]$ . The goal of the control policy is to steer the retransmission traffic in the direction of the optimum. Consequently the choice of  $\gamma, \beta$ , and  $a(0), a(1)$  and  $a(e)$  is closely associated with the dynamics of the traffic intensity. In order to obtain  $a(0), a(1), a(e)$ , we construct a local model as in [15]. For an integer  $n > 0$ , the Markov process  $(f_t)$  obtained by localizing the Markov process  $(N_t, f_t)$  to  $(N_t) = n$  evolves as follows.

$$\hat{f}_{t+1} = \{\beta \wedge \alpha^\gamma(Z_t) \hat{f}_t\} \tag{3.2}$$

If we define  $\hat{\Psi}_t = \ln(n \hat{f}_t)$  for  $n > 0$ , then  $(\hat{\Psi}_t)$  is also a Markov process that evolves as

$$\hat{\Psi}_{t+1} = \{\ln(n\beta) \wedge \hat{\Psi}_t + \gamma c(Z_t)\} \tag{3.3}$$

where  $c(i) = \ln(a(i))$  for  $i = 0, 1, e$ . Note that the transitions of  $\hat{\Psi}_t$  do not depend on  $n$  except through the term  $\ln(n\beta)$ . Hence we will study the following stochastic recursion algorithm by assuming that  $\Psi_t \leq \ln(n\beta)$  for all  $t$  (this assumption will be less crucial and can be justified by increasing  $n$  [15])

$$\hat{\Psi}_{t+1} = \hat{\Psi}_t + \gamma c(Z_t) \tag{3.4}$$

#### 2. Choice of Parameters

The choice of the retransmission control parameters will be made based on the drift of the local model. Lemma(4.3) in section II' justifies this approach by establishing that the error approximating the global drift by the local drift goes to zero as the backlog grows. Define

$$\hat{m}(\Psi) = \gamma^{-1} E[\Delta \hat{\Psi}_t | \hat{\Psi}_t = \Psi] \tag{3.5}$$

and

$$\hat{v}(\Psi) = \gamma^{-2} E[(\Delta \hat{\Psi}_t)^2 | \hat{\Psi}_t = \Psi] - \hat{m}^2(\Psi) \tag{3.6}$$

where  $\Delta \hat{\Psi}_t = \hat{\Psi}_{t+1} - \hat{\Psi}_t$ . Then  $\hat{m}(\Psi)$  and  $\hat{v}(\Psi)$  are the normalized drift and variance functions of the real-valued Markov process  $(\Psi_t)$ .

In order for  $\Psi_t$  to move toward a stable value  $\Psi^*$ , we need to choose  $c(0)$ ,  $c(1)$ ,  $c(e)$  so that

$$\text{sgn}(G(x) - G^*) = -\text{sgn}(\hat{m}(G(x))) \quad \forall x \in R \quad (3.7)$$

where  $\text{sgn}(y) = 1$  for  $y > 0$  and  $0$  for  $y = 0$  and  $-1$  for  $y < 0$ .

There are many choices of  $c(1)$  but we set  $c(1) = 0$ . After normalization, we find that

$$c_0 = c(0) = \frac{1 - (\theta G^* e^{-\theta G^*} + e^{-\theta G^*})}{1 + e^{-\theta G^*} - (\theta G^* e^{-\theta G^*} + e^{-\theta G^*})} \quad (3.8)$$

$$c_1 = c(1) = 0 \quad (3.9)$$

$$c_e = c(e) = -\frac{e^{-\theta G^*}}{1 + e^{-\theta G^*} - (\theta G^* e^{-\theta G^*} + e^{-\theta G^*})} \quad (3.10)$$

With this choice, it follows from the definition that

$$\hat{m}(\Psi) = c_e e^{-\theta \Psi} + c_e [1 - (\theta G e^{-\theta \Psi} + e^{-\theta \Psi})] \quad (3.11)$$

where  $G = \lambda + e^*$ .

$\hat{m}(\varphi)$  has the following properties and will be used to prove the stability of the global model. LEMMA(3.1)

(i) for any  $\lambda$  and any  $0 < \theta \leq 1$ , the function  $\hat{m}(\Psi)$  is strictly decreasing in  $\Psi$ .

(ii) for any  $0 < \theta \leq 1$ , there exists a unique  $\Psi^* \in R$  such that  $\hat{m}(\Psi^*) = 0$ .

PROOF: (i) This is simply implied by the inequality

$$\frac{\partial}{\partial \Psi} \hat{m}(\Psi) = G' \{-c_0 e^{-\theta \Psi} + c_e [\theta G e^{-\theta \Psi} + \theta(e^{-\theta \Psi} - e^{-\theta \Psi})]\} \quad (3.12)$$

because  $G' = e^\Psi > 0$  and  $e^{-\theta \Psi} - e^{-\theta \Psi} \geq 0$  for  $0 < \theta \leq 1$ , and  $c_0 > 0$  and  $c_e < 0$ .

(ii) Existence of a solution in  $R$  follows from  $\hat{m}(+\infty) < 0$  and  $\hat{m}(-\infty) > 0$  and the continuity of

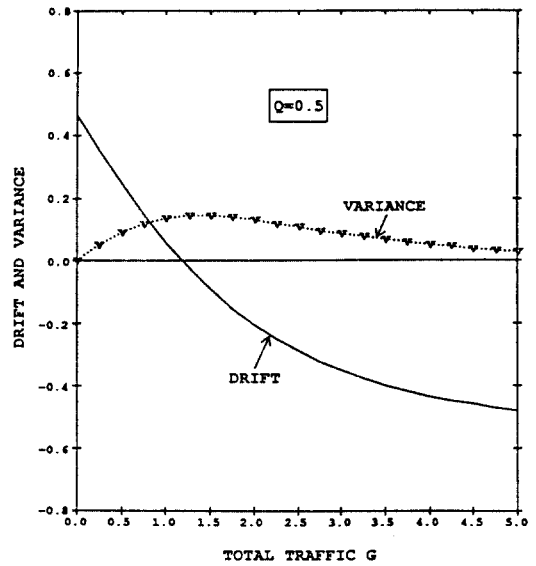


Fig.1. Drift and variance functions as a function of total traffic intensity for  $Q = 0.5$ .

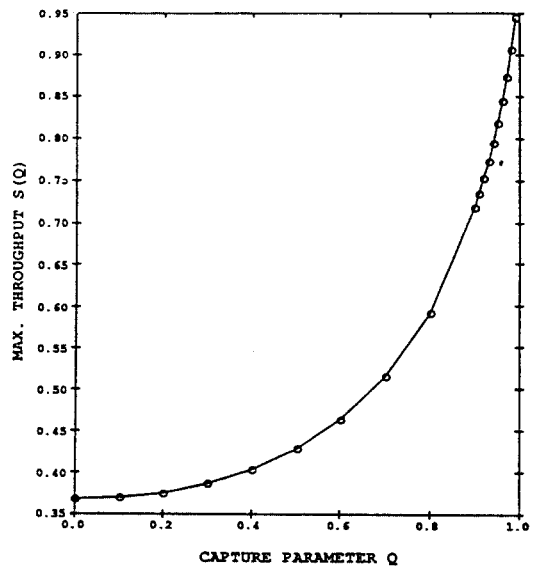


Fig.2. Maximum achievable channel throughput versus  $Q$ .

$\hat{m}(\Psi)$ . And uniqueness comes from the strict monotonicity of  $\hat{m}(\Psi)$ . □

A typical drift function is plotted in Fig.1 and shows the existence of globally stable points for  $Q=0.5$ . The variance function  $\hat{v}(G)$  is also plotted in Fig.1.

Having specified the retransmission control policy we now compute the expected throughput. We first approximate the achievable throughput and then relate it to the actual throughput.

Recall that the global throughput

$$S(n, f) = \sum_{j=0}^{\infty} \sum_{m=0}^n \frac{\lambda^j e^{-\lambda}}{j!} \binom{n}{m} f^m (1-f)^{n-m} C_{j+m}.$$

Then using Poisson approximations to Binomial, we may approximate  $S(n, f)$  by

$$S(\mu) = \bar{S}(G) = \theta G e^{-G} + e^{-\theta G} - e^{-G} \tag{3.13}$$

where with  $\mu = nf$ , and  $G = \lambda + \mu$ . (3.14)

As  $Q$  moves toward unity both  $G^*$  and  $S^*$  increase exponentially towards 5 and 1 (see Fig. 2), respectively, and when  $\theta=1$ , we get  $G^*=1$  at which  $S^*=e^{-1}$ , the famous number for the slotted ALOHA system with no capture.

#### IV. STABILITY ANALYSIS

In this section we prove that the global model (not the local Poisson approximation) is stable by showing the geometric ergodicity of the Markov chain  $X_t = (N_t, f_t)$ . We use a technique established by Hajek [19]. This approach was first used by Cruz [18] to study an ALOHA system in which the feedback channel is prone to errors, and by Tsitsikis [17] to study a pseudo-Bayesian broadcast control scheme to study an ALOHA system in which transmission errors as well as a certain form of capture may occur.

If  $X$  and  $Y$  are random variables, then  $Y$  is said to *stochastically dominate*  $X$ , written  $X < Y$ , if  $P[X$

$$> c] \leq P[Y > c]$$
 for  $c \in R$ .

A random variable  $Z$  is said to be *exponential type* if there exist  $s > 0, D$ , such that

$$E[e^{sZ}] \leq D$$

Let  $\{Y_t, t \geq 0\}$  be a sequence of random variables on a probability space  $(Q, F, P)$  adapted to an increasing sequence  $\{F_t, t \geq 0\}$  of sub  $\sigma$ -field of  $F$ , thus  $Y_t$  is  $F_t$ -measurable for each  $t$ , then the sequence  $\{Y_t, F_t\}$  is said to be *exponential type* if there exist  $s > 0, D$ , such that

$$E[e^{s(Y_{t+1} - Y_t)} | F_t] \leq D \quad \forall t \geq 0$$

Let  $\{X_t\}$  be an irreducible aperiodic Markov chain on a denumerable state space, then  $\{X_t\}$  is *geometrically ergodic*, if the stopping time  $\tau = \min\{t > 0 : X_t = x_0\}$  is exponential type for some initial state  $X_0 = x_0$ .

The proof of stability is based on a result due to Hajek [19]. We reproduce Hajek's result for convenience.

#### PROPOSITION (Hajek)

Suppose that  $\{W_t, F_t\}$  is exponential type and that for some  $\epsilon > 0, a \in R$ , we have

$$E[W_{t+1} - W_t : W_t > a | F_t] \leq -\epsilon \quad \forall t \geq 0.$$

Then the stopping time  $\tau = \min\{t \geq 0 : W_t \leq a\}$  is exponential type provided that  $W_0$  is exponential type.

Assume, for some relatively prime integer  $i, j$ ,  $a' = (v^i, 1, v^j)$  where  $v = v(\theta)$  such that  $0 < v < 1$  and  $0 < \theta < 1$ . Then we may consider the process  $X_t = (N_t, f_t)$  as a Markov chain with the denumerable state space,  $\{(n, \beta v^k) : n, k \text{ nonnegative integers}\}$ .

#### THEOREM (4.1)

if  $\lambda < S^*$  and  $\gamma < \gamma^*$  for some  $\gamma^*$ , then the Mar-

kov chain  $X_t = (N_t, f_t)$  is geometrically ergodic and  $\limsup_t E[N_t] < B$  for some  $B < \infty$

□

In order to prove this, we introduce a continuous Lyapunov function. Let

$$W_t \equiv W(N_t, V_t) = N_t + \alpha g_\Delta(V_t) \text{ for some } \alpha, \Delta > 0 \tag{4.0}$$

where

$$V_t = \Psi_t - \Psi^* \text{ and } \Psi_t = \ln [(N_t \vee 1) f_t], \tag{4.1}$$

$$g_\Delta(x) = \Delta^2 \ln \cosh\left(\frac{x}{\Delta}\right) \quad \forall x \in R, \tag{4.2}$$

Note that with  $X_t = (N_t, V_t)$ ,  $W(X_t)$  is nonnegative and continuous in the open space

$$S_b = \{X_t : W(X_t) < b \text{ for any } b > 0\}.$$

Both Cruz[18], and Schwart and Sidi[5] have employed this approach but with a different Lyapunov function which is composed of two connected functions. Note that the function we proposed resolves the discontinuity problem as in [5] and [18], and that it may be utilized to prove the stability for the higher dimensional Markovian systems.

LEMMA (4.1)

$$\text{Let } f(x) = -\frac{1}{2} \Delta^{-2} g_\Delta^{(3)}(\Delta x) = \tanh(x) \sec h^2(x).$$

Then

$$\begin{aligned} f^{(n)}(x) = & -\binom{n}{0} \tanh(x) f^{(n-1)}(x) \\ & -\binom{n}{1} \sec h^2(x) f^{(n-2)}(x) \\ & +\binom{n}{2} f(x) f^{(n-3)}(x) + \dots \\ & +\binom{n}{n-2} f^{(1)}(x) f^{(n-4)}(x) \\ & -\binom{n}{n-1} f(x) f^{(n-3)}(x) \\ & -\binom{n}{n} \sec h^2(x) f^{(n-2)}(x). \end{aligned}$$

PROOF : Let  $u(x) = \tanh(x)$  and  $v(x) = \sec h^2(x)$ . Then by Leibnitz formula,

$$f^{(n)}(x) = (uv)^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(k)}(x) v^{(n-k)}(x).$$

Note that

$$\begin{aligned} u^{(n)}(x) &= -2f^{(n-2)}(x), \\ v^{(n)}(x) &= -2f^{(n-1)}(x). \end{aligned}$$

Substituting these into the above equation, we have the result.

□

In view of the fact that  $u(x), v(x)$  are continuous and finite  $\forall x \in R$ , this result shows that  $f^{(n+1)}(x)$  exists and is finite  $\forall x \in R$ .

Based on Lemma(4.1) and Taylor's theorem, we may obtain the following expansion,

$$g_\Delta(x_2) - g_\Delta(x_1) = \sum_{k=1}^{n-1} \frac{g_\Delta^{(k)}(x_1)}{k!} (x_2 - x_1)^k + R_n(\eta) \tag{4.3}$$

where

$$\begin{aligned} R_n(\eta) &= \frac{g_\Delta^{(n)}(\eta)}{(n)!} (x_2 - x_1)^n, \text{ and } \eta = \kappa x_1 + (1 - \kappa)x_2 \\ &\text{for some } 0 < \kappa < 1. \end{aligned}$$

Now if we let  $h^{(n)}(x) = \Delta^{-2} g_\Delta^{(n)}(x) \quad \forall n \geq 3$ , then

$$R_n(x) = \frac{h^{(n)}(x)}{\Delta^{n-2} n!} (x_2 - x_1)^n \quad \forall n \geq 3 \tag{4.4}$$

and

$$h^{(n)}(x) < \infty \quad \forall n \geq 3 \tag{4.5}$$

Note that

$$0 \leq |g_\Delta'(x)| \leq \Delta \quad \forall x, \tag{4.6}$$

$$0 \leq g_\Delta''(x) \leq 1 \quad \forall x, \tag{4.7}$$

$$0 \leq |g_\Delta'''(x)| \leq \frac{2}{\Delta} \quad \forall x. \tag{4.8}$$

Using power series expansions we can also establish that

$$\ln \cosh(x) = \frac{x^2}{2} + O(x^4) \text{ for } |x| < 1 \quad (4.9)$$

and

$$\ln \cosh(x) = |x| - \ln 2 + O(e^{-2|x|}) \text{ for } |x| > 1 \quad (4.10)$$

Let  $Y_t$  be Poisson process with mean  $\lambda$ . Then the drift of the channel backlog is always bounded above by  $\lambda$ .

$$E [ N_{t+1} - N_t | F_t ] \leq \lambda \quad (4.11a)$$

The following result, which is tighter than Eq. (4.11a), will be used for proving that  $\{W_t, F_t\}$  is exponential type. We know that

$$N_{t+1} = N_t + Y_t - I(Z_t = 1)$$

So

$$\begin{aligned} E [ N_{t+1} - N_t | F_t ] &= \lambda - E [ I(Z_t = 1) | F_t ] \\ &= \lambda - P_1 \\ &= \lambda - \hat{P}_1 + O(N_t^{-1}) \end{aligned} \quad (4.11b)$$

The last two steps may be justified from the definitions of  $P_1$  and  $\hat{P}_1$  and using Proposition (A. 2) in Appendix A. □

The first of the two conditions of Hajek's Proposition to be satisfied can now be established.

**PROPOSITION (4.1)**

For some positive and finite values  $L_0$  and  $L_1$ ,

$$( | W_{t+1} - W_t | | F_t ) < L_0 + L_1 Y_t$$

**PROOF :** From the definition of  $\Psi_t$  in Eq. (4.1).

$$\Psi_{t+1} - \Psi_t = \ln \left[ \frac{(N_{t+1} \vee 1) f_t}{(N_t \vee 1)} \right]$$

$$+ \ln \left[ \frac{\alpha^\gamma(Z_t) f_t \wedge \beta}{f_t} \right]$$

and thus

$$| \Psi_{t+1} - \Psi_t | \leq \frac{1}{N_t \vee 1} | N_{t+1} - N_t | + \gamma | C(Z_t) | \quad (4.12)$$

Since  $| N_{t+1} - N_t | < Y_t + 1$

$$\begin{aligned} &\leq \frac{Y_t + 1}{N_t \vee 1} + \max_i \gamma | C_i | \\ &\leq Y_t + 1 + \max_i \gamma | C_i | \end{aligned} \quad (4.13)$$

Hence

$$( | \Psi_{t+1} - \Psi_t | | F_t ) < \tau_0 + Y_t \text{ where } \tau_0 = 1 + \max_i \gamma | C_i | \quad (4.14)$$

Next by mean value theorem, there is some  $y \in [x_1, x_2]$  such that

$$g_\Delta(x_2) = g_\Delta(x_1) + g_\Delta'(y)(x_2 - x_1)$$

Because  $| g_\Delta'(x) | \leq \Delta \forall x$ ,

$$| g_\Delta(x_2) - g_\Delta(x_1) | \leq \Delta | x_2 - x_1 | \quad (4.15)$$

Finally, by combining Eqs.(4.14) and (4.15), we get

$$\begin{aligned} | W_{t+1} - W_t | &\leq | N_{t+1} - N_t | + \alpha | g_\Delta(I_{t+1}) - g_\Delta(I_t) | \\ &\leq Y_t + 1 + \alpha \Delta (\tau_0 + Y_t), \end{aligned}$$

which leads to the desired result with  $L_0 = \alpha \Delta \tau_0 + 1$  and  $L_1 = \alpha \Delta + 1$ . □

The above Proposition implies that  $\{W_t, F_t\}$  is exponential type. It remains for us to prove that the drift of  $W_t$  is negative. This will be done in several stages.

In Lemma (4.2) we bound the  $n$ -th moment of the drift of the retransmitted traffic intensity.



LEMMA (4.2)

$$E[(\Psi_{t+1} - \Psi_t)^n | F_t] = O(N_t^{-n}) + O(\gamma^n) \quad \forall n < \infty$$

PROOF: Using the fact that  $(a+b)^n \leq 2^{n-1}(a^n + b^n)$  for  $n \geq 2$  and from Eq. (4.13),

$$E[(\Psi - \Psi_t)^n | F_t] \leq 2^{n-1} \left\{ \frac{E[(Y_t + 1)^n | F_t]}{N_t^n} + \gamma^n \max |c_i|^n \right\}.$$

Since  $Y_t$  is Poisson random variable, all moments exist and so

$$E[Y_t^n] < \infty \quad \forall n < \infty. \quad \square$$

Lemma (4.2), Eq.(4.3) and the fact that

$$|R_n(\eta_t)| < O(\Delta^{n-2}) E[(\Psi_{t+1} - \Psi_t)^n | F_t] \quad \forall n \geq 3 \quad (4.16)$$

where  $\eta_t = \kappa V_t + (1-\kappa)V_{t+1}$ , imply the following result

$$\begin{aligned} E[g_\Delta(V_{t+1}) - g_\Delta(V_t) | F_t] &= \Delta \tanh \left[ \frac{V_t}{\Delta} \right] E[\Psi_{t+1} - \Psi_t | F_t] \\ &+ \sec^2 \left[ \frac{V_t}{\Delta} \right] E[(\Psi_{t+1} - \Psi_t)^2 | F_t] \\ &+ O(\Delta^{-1})O(N_t^{-3}) + O(\Delta^{-1})O(\gamma^3). \end{aligned} \quad (4.17) \quad \square$$

In the following Lemma, the drift function of the retransmitted traffic intensity is approximated by  $\hat{m}(\Psi_t)$ , obtained by using Poisson approximations to Binomial probabilities. It is shown that the error is  $O(N_t^{-1})$ .

LEMMA (4.3)

$$E[\Psi_{t+1} - \Psi_t | F_t] = \gamma \hat{m}(\Psi_t) + O(N_t^{-1}) + O(\gamma).$$

PROOF: Using Eq. (4.12),

$$\begin{aligned} |E[\Psi_{t+1} - \Psi_t | F_t] - \gamma \hat{m}(\Psi_t)| &\leq \left| \frac{1}{N_t \vee 1} E[|N_{t+1} - N_t| | F_t] \right. \\ &\quad \left. + \gamma \{E[c(Z_t) | F_t] - \hat{m}(\Psi_t)\} \right| \quad (4.18) \\ &\leq \frac{\lambda}{N_t \vee 1} + \gamma \{c_0 |P_0 - P_0| + |c_e| |P_e - P_e|\}. \quad (4.19) \end{aligned}$$

Eq.(4.19) follows from Eq.(4.11a) and the definition of  $C(Z_t)$  and  $c_1=0$ . So the result follows from Propositions (A.1) and (A.3) in Appendix A. The possibility that the inequality of Prop. 4.1 is strict is a direct consequence of the possibility that truncation may occur. □

The following approximation for the drift of  $W_t$  can now be derived based on Eqs. (4.0), (4.7), (4.17) and Lemma (4.2) and (4.3).

$$E[W_{t+1} - W_t | F_t] = E[N_{t+1} - N_t | F_t] + \alpha \{g'_\Delta(V_t) \{ \gamma \hat{m}(\Psi_t) + O(N_t^{-1}) \} + O(N_t^{-2}) + O(\gamma^2)\} \quad (4.20)$$

Note that from Lemma (3.1) in section III and the fact that  $g'_\Delta(x) = \Delta \tanh\left(\frac{x}{\Delta}\right)$  it follows that

$$g'_\Delta(V_t) \hat{m}(\Psi_t) \leq 0 \quad \forall V_t \in R. \quad (4.21)$$

This is a key fact.

Now we prove the negative drift of  $W_t$ . If  $a = a_1 + a_2$ , then in order for  $W > a$  either  $N > a_1$  or  $\alpha g'_\Delta(V) > a_2$  because  $W = N + \alpha g_\Delta(V)$ . Let  $a_2 = \alpha \Delta^2 \text{Incosh}(1)$ . Then, the following are equivalent events

$$\{\alpha g'_\Delta(V) > a_2\} \Leftrightarrow \{|V| > \Delta\}. \quad (4.22)$$

PROPOSITION (4.2)

For some  $\epsilon > 0$ ,  $a \in R$ , we have

$$E[W_{t+1} - W_t : W_t > a | F_t] \leq -\epsilon \quad \forall t \geq 0.$$

PROOF: To prove this, four possible regions should

be considered. Note that  $X_t = (N_t, V_t)$  has an equivalent representation as  $(N_t, \Psi_t)$  or  $(N_t, f_t)$  since they are instantaneous functions of both  $N_t$  and  $f_t$ .

Given some  $a_1, \delta < 1$  and  $\Delta \in (\delta, \infty)$  we partition  $R^2$  of the state space into four regions as follows. Let

$$Q_1 = \{X_t : N_t > a_1 \text{ and } |V_t| < \delta\},$$

$$Q_2 = \{X_t : N_t > a_1 \text{ and } |V_t| < \delta\},$$

$$Q_3 = \{X_t : 0 \leq N_t \leq a_1 \text{ and } |V_t| > \Delta\},$$

$$Q_4 = \{X_t : N_t > a_1 \text{ and } |V_t| > \Delta\}.$$

Note that

$$Q_4 \subset Q_2$$

Hence we need to consider only Q1, Q2 and Q3.

CASE 1 :  $X_t \in Q_1$

In this case,  $N_t$  is large and  $|V_t|$  is small. From Proposition (B.1) in Appendix B, we draw the following consequence since  $\delta$  is chosen small :

$$\hat{P}_1 = S^* + O(\delta). \tag{4.23}$$

Using Eq (4.11b) and combining with Eq.(4.23), we have

$$\begin{aligned} E[N_{t+1} - N_t | F_t] &= \lambda - \hat{P}_1 + O(N_t^{-1}) \\ &= \lambda - S^* + O(\delta) + O(N_t^{-1}). \end{aligned}$$

Consequently, if we choose  $\delta$  sufficiently small and  $a_1$  large enough, there exists  $\epsilon > 0$  such that if  $\lambda < S^*$ , then

$$\begin{aligned} E[N_{t+1} - N_t | F_t] &= \lambda - S^* + O(\delta) + O(N_t^{-1}) \leq -2\epsilon \\ &\text{for arbitrarily small } \epsilon > 0. \end{aligned}$$

Note also that from the properties of  $g_\Delta'(V_t)$  and  $\hat{m}(\Psi_t)$

$$g_\Delta'(V_t) \hat{m}(\Psi_t) \uparrow 0 \text{ as } \delta \downarrow 0,$$

however,  $g_\Delta'(V_t) \hat{m}(\Psi_t)$  is still a negative quantity. Furthermore, if  $N_t$  is large enough and  $\gamma$  is small enough,  $N_t > a_1'$  for some  $a_1'$  and  $\gamma < \gamma'$  for some  $\gamma' < 1$  implies

$$\begin{aligned} \gamma g_\Delta'(V_t) \hat{m}(\Psi_t) + g_\Delta'(V_t) O(N_t^{-1}) + O(N_t^{-2}) + O(\gamma^2) \\ \leq g_\Delta'(V_t) O(N_t^{-1}) + O(N_t^{-2}) + O(\gamma^2) \leq \frac{\epsilon}{\alpha} \text{ for any} \\ \alpha \geq 1 \text{ and } \epsilon > 0. \end{aligned}$$

Hence from Eq.(4.20), if  $W_t > a'$  for some large  $a'$  and  $\gamma < \gamma'$  we have

$$\begin{aligned} E[W_{t+1} - W_t | F_t] &\leq E[N_{t+1} - N_t | F_t] \\ &+ \alpha \gamma g_\Delta'(V_t) \hat{m}(\Psi_t) + g_\Delta'(V_t) O(N_t^{-1}) + O(N_t^{-2}) + O(\gamma^2); \\ &\leq -2\epsilon + \alpha \leq \frac{\epsilon}{\alpha} \leq -\epsilon \quad \forall X_t \in Q_1. \end{aligned} \tag{4.24}$$

CASE 2 :  $X_t \in Q_2$ .

Since  $|V_t| \geq \delta$ , we have  $|g_\Delta'(V_t)| \geq \delta$  for  $0 < \delta < 1$  (see Eqs.(4.9) and (4.10) and also we can find  $\sigma > 0$  such that

$$|\hat{m}(\Psi_t)| > \sigma. \tag{4.25}$$

Hence  $g_\Delta'(V_t) \hat{m}(\Psi_t) \leq -\delta\sigma$ . Now if we choose  $N_t$  sufficiently large and  $\gamma$  small, there exist some  $a_1''$  and  $\gamma''$  such that  $N_t > a_1''$  and  $\gamma < \gamma''$  imply that

$$\begin{aligned} \gamma g_\Delta'(V_t) \hat{m}(\Psi_t) + g_\Delta'(V_t) O(N_t^{-1}) + O(N_t^{-2}) + O(\gamma^2) \\ \leq -\xi \text{ for some } \xi > 0. \end{aligned}$$

Therefore, we have the desired result by choosing  $\alpha \geq \frac{\lambda + \epsilon}{\xi}$ , that is, if  $W_t > a''$  for some  $a''$  and  $\gamma < \gamma''$ , we have from Eq. (4.20)

$$\begin{aligned} E[W_{t+1} - W_t | F_t] &= E[N_{t+1} - N_t | F_t] \\ &+ \alpha \gamma g_\Delta'(V_t) \hat{m}(\Psi_t) + g_\Delta'(V_t) O(N_t^{-1}) + O(N_t^{-2}) + O(\gamma^2); \\ &\leq \lambda - \alpha \xi \leq \lambda - \frac{(\lambda + \epsilon)}{\xi} \leq -\epsilon \quad \forall X_t \in Q_2. \end{aligned} \tag{4.26}$$

CASE 3 :  $X_t \in Q_3$ .

Note that if  $V_t$  is positive and very large, then  $N_t$  should be large because  $V_t = \Psi_t - \Psi^*$  and  $0 \leq f_t \leq 1$ . Thus this case is equivalent to CASE 2 by choosing  $\Delta$  sufficiently large.

Now let us choose  $\Delta$  large enough so that  $|V_t|$  is very large. If  $N_t \leq a_1$  (that is,  $N_t$  is not large) but  $|V_t|$  is very large, then  $f_t$  should be very small. Therefore  $\{X_t \in Q_3\}$  is equivalent to  $\{V_t < -\Delta\}$ . In this case, the terms  $O(N_t^{-1})$  and  $O(N_t^{-2})$  in Eq.(4.20) are bounded, but can not be made as small as we wish because  $N_t \leq a_1$ . However, since  $V_t < -\Delta$ , we have  $g_\Delta'(V_t) \leq -\Delta \tanh(1)$  and so  $g_\Delta'(V_t) \hat{m}(\Psi_t) < -\sigma \Delta \tanh(1)$  for some  $\sigma > 0$ . Thus choosing  $\Delta$  sufficiently large and  $\gamma$  small, we have for any  $0 < N_t < a_1$

$$\gamma g_\Delta'(V_t) \hat{m}(\Psi_t) + g_\Delta'(V_t) O(N_t^{-1}) + O(N_t^{-2}) + O(\gamma^2) \leq -\frac{\lambda + \epsilon}{\alpha}$$

for any  $\alpha > 0$  and some  $\epsilon > 0$ .

Hence from Eq.(4.20), there exist some  $a'''$  and  $\gamma'''$  such that if  $W_t > a'''$  and  $\gamma < \gamma'''$ ,

$$E[W_{t+1} - W_t | F_t] \leq -\epsilon \quad \forall X_t \in Q_3. \tag{4.27}$$

The proof of the main theorem may now be completed. Let  $a = \max\{a', a'', a'''\}$  and  $\gamma^* = \min\{\gamma', \gamma'', \gamma'''\}$ . Finally, if  $W(N_t, V_t) \geq a$ , and  $\gamma < \gamma^*$ , then from Eqs.(4.24), (4.26) and (4.27)

$$E[W_{t+1} - W_t | F_t] \leq -\epsilon \quad \forall X_t \in Q_1 \cup Q_2 \cup Q_3. \tag{4.28}$$

From Proposition (4.1),  $\{W_t, F_t\}$  is known to be exponential type. Hence by Proposition of Hajek we know that the stopping time  $\tau = \min\{t > 0 : W_t(N_t, V_t) < a\}$  is exponential type, for any initial state of exponential type. From this, it follows that the time until  $(N_t, V_t)$  becomes equal to  $(0, 0)$  is also exponential type, and thus chain  $X_t = (N_t, f_t)$  is geometrically ergodic. Furthermore, Propositions (4.1) and (4.2) imply  $\sup_t E[W_t] < B_1$  for some  $B_1 < \infty$  and so  $\sup_t E[N_t] < B$  for

some  $B < \infty$  because  $N_t \leq W_t$  for all  $t$ . Now the conclusion of Theorem (4.1) is completed. □

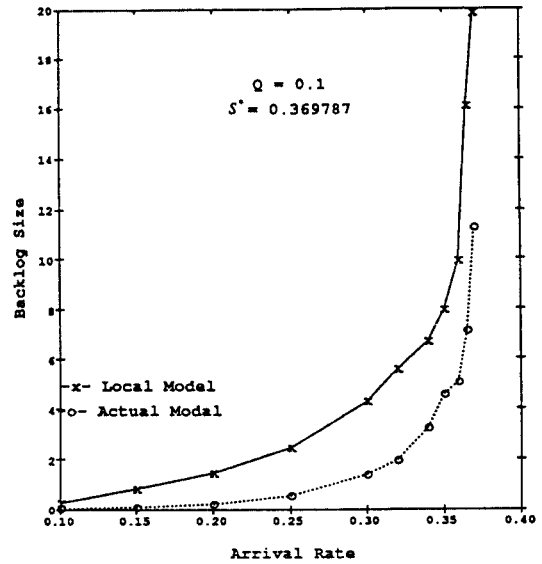


Fig.3. Average backlog versus mean arrival rate for  $Q = 0.1$  and  $\gamma = 0.3$ .

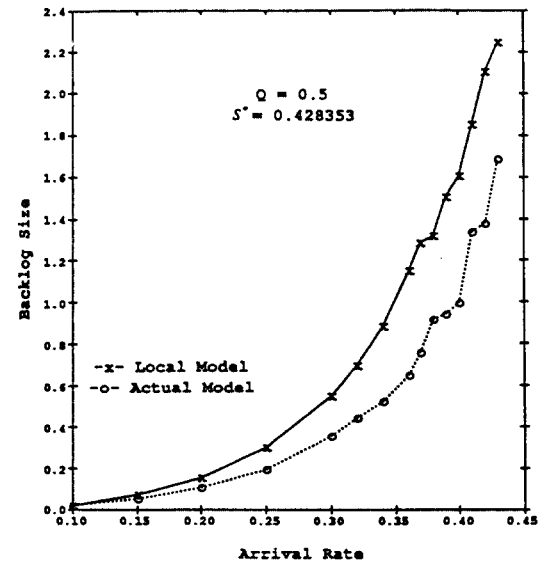


Fig.4. Average backlog versus mean arrival rate for  $Q = 0.5$  and  $\gamma = 0.3$ .

## V. SIMULATION RESULTS

The purpose of this section is to demonstrate, by simulation, that the slotted ALOHA system with capture is stable for any input rate below the maximum achievable channel throughput when some retransmission control algorithm is employed.

The system was simulated for 50,000 time slots for each choice of arrival rate  $\lambda$ . For each trial the sample mean and the sample variance and the maximum backlog size were obtained during the 50,000 time slots. Throughout, we used  $\gamma = 0.3$  which was suggested as the best choice for the slotted ALOHA system with no capture [15]. We tested this by varying  $\gamma$  for typical values of  $Q$ . A typical result is shown in Fig.3 and indicates that when the capture parameter increases the system performance is insensitive to  $\gamma$ . Based on our simulations we conclude that a good range of  $\gamma$  is (0.2, 0.6). In Figs.3-4, we plot the average backlog size versus the mean arrival rate for the different values of  $\gamma$  and we show that the mean backlog is well bounded when operating below the maximum possible input rate. We note that when  $Q = 0.1$  the improvement in throughput is noticeable and as  $Q$  increases the improvement is dramatic.

The simulation results suggest the following :

- (1) The control algorithm considered stabilizes the system for any input rate below the channel capacity.
- (2) The difference between the actual model and the local Poisson approximation is within some permissible error in terms of the average backlog size.
- (3) For a given input rate, the average backlog using local Poisson approximation is generally decreasing larger than that of the actual model. As  $Q$  increases the distinction is less noticeable.
- (4) As the capture parameter  $Q$  increases, the mean backlog in overall range is generally decreasing for any input rate below the capacity.

- (5) The mean backlog is not sensitive to changes in  $\gamma$ , but it gets larger when  $\gamma$  is either very small or very large. Good choices for  $\gamma$  range from about 0.2 to about 0.6.

We conclude based on the simulation that capture reduces the delay dramatically, as well as improves the channel utilization remarkably, compared to the classical ALOHA without capture.

## VI. CONCLUSION

We have studied a slotted capture ALOHA system. In view of the instability of the system for a class of capture probabilities, it was necessary to use retransmission control policies. We stabilize the system by employing a retransmission control algorithm and can derive the achievable channel capacity as a function of the capture parameter. We prove the stability of the multidimensional Markovian model by using a continuous Lyapunov function which may be applicable for other complex multidimensional systems. We also have obtained a delay performance through computer simulation to show the stability for any input rate below the maximum achievable channel throughput. Based on the simulation, capture reduces the delay dramatically, as well as improves the channel utilization remarkably, compared to the classical ALOHA without capture.

### APPENDIX A

In this Appendix, we establish Propositions (A.1) (A.3).

$$P_j = P[ Z_t = j | N_t = n, f_t = f ]$$

and

$$\hat{P}_j = P[ Z_t = j | \hat{\Psi}_t = \Psi ]$$

Then it can be shown using the channel transition probabilities with  $\theta = 1 - Q$ ,

$$P_0 = e^{-\lambda}(1-f)^n \tag{A.1}$$

$$P_1 = \theta e^{-\lambda} n f (1-f)^{n-1} + \theta \lambda e^{-\lambda} (1-f)^n + (1-\theta f) e^{-\lambda \theta} - e^{-\lambda} (1-f)^n \tag{A.2}$$

and

$$P_e = 1 - [\theta e^{-\lambda} n f (1-f)^{n-1} + \theta \lambda e^{-\lambda} (1-f)^n + (1-\theta f)^n e^{-\lambda \theta}]. \tag{A.3}$$

Now the approximate version,  $\hat{P}_j$ , of  $P_j$  can be expressed as follows after using the local Poisson approximation[15].

$$\hat{P}_0 = e^{-G}, \tag{A.4}$$

$$\hat{P}_1 = (\theta G - 1)e^{-G} + e^{-\theta G}, \tag{A.5}$$

$$\hat{P}_e = 1 - [\theta G e^{-G} + e^{-\theta G}] \tag{A.6}$$

where  $G = \lambda + e^*$ . Following the approach of [18], We show that the errors of these approximations converge uniformly to zero if either  $n \rightarrow \infty$  or  $f \rightarrow 0$ .

PROPOSITION (A.1)

$$|P_0 - \hat{P}_0| \leq \frac{8e^{-2}}{n} \tag{A.7}$$

$$|P_0 - \hat{P}_0| \leq 2e^{-1}f \tag{A.8}$$

PROOF :

$$|P_0 - \hat{P}_0| \leq e^{-\lambda} |(1-f)^n - e^{-nf}| \leq \frac{8e^{-2}e^{-\lambda}}{n} \leq \frac{8e^{-2}}{n}.$$

and

$$|P_0 - \hat{P}_0| \leq 2e^{-1}e^{-\lambda}f \leq 2e^{-1}f$$

□

PROPOSITION (A.2)

$$|P_1 - \hat{P}_1| \leq \frac{1}{n} [16e^{-2} + 62\theta e^{-3}], \tag{A.9}$$

$$|P_1 - \hat{P}_1| \leq f [10\theta e^{-2} + 2(1+\theta)e^{-1}] \tag{A.10}$$

PROOF : By the definitions of  $P_1$  and  $\hat{P}_1$ ,

$$\begin{aligned} |P_1 - \hat{P}_1| &\leq \theta e^{-\lambda} |nf(1-f)^{n-1} - nfe^{-nf}| \\ &\quad + |e^{-\lambda} + \lambda\theta e^{-\lambda}| |e^{-nf} - (1-f)^n| \\ &\quad + e^{-\lambda\theta} |(1-\theta f)^n - e^{-\theta nf}| \\ &\leq \frac{1}{n} [54\theta e^{-3}e^{-\lambda} + 8e^{-2}(e^{-\lambda} + e^{-\lambda\theta} + \theta\lambda e^{-\lambda})] \end{aligned}$$

using the fact that  $e^{-\lambda} \leq e^{-\lambda\theta} \leq 1$  and  $\lambda e^{-\lambda} \leq e^{-1}$ ,

$$\leq \frac{1}{n} [16e^{-2} + 62\theta e^{-3}].$$

This is (A.9). Similarly, (A.10) can be shown. □

PROPOSITION (A.3)

$$|P_e - \hat{P}_e| \leq \frac{1}{n} [24e^{-2} + 62\theta e^{-3}],$$

$$|P_e - \hat{P}_e| \leq f [10\theta e^{-2} + 2(2+\theta)e^{-2}].$$

PROOF : Noting that

$$\begin{aligned} |P_e - \hat{P}_e| &= |\hat{P}_0 - P_0 + \hat{P}_1 - P_1| \\ &\leq |P_0 - \hat{P}_0| + |P_1 - \hat{P}_1| \end{aligned}$$

and using Proposition (A.1)-(A.2), one can have the desired results. □

### APPENDIX B

In this appendix, we prove the following Proposition. Recall that

$$V_t = \Psi_t - \Psi^* = \ln \mu_t - \ln \mu^* \tag{B.1}$$

where  $\mu^*$  is the optimal value of  $\mu_t$ , which depends on the capture parameter  $\theta$ , and

$$\hat{P}_1 = (\theta G - 1)e^{-G} + e^{-\theta G}, \quad (B.2)$$

$$S^* = (\theta G^* - 1)e^{-G^*} + e^{-\theta G^*}. \quad (B.3)$$

PROPOSITION (B.1)

If  $|V_t| < \delta < 1$ , then

$$|S^* - \hat{P}_1| \leq A\delta \text{ for some } A > 0.$$

To prove this, we give a Lemma in what follows.

Let

$$d(x, y) = |x - y| \text{ for some } x, y \in R.$$

Then if  $|V_t| < \delta$ , we may have from (B.1)

$$d(\mu^*, \mu) < \max\{\mu^*|1 - e^{-\delta}|, \mu^*|1 - e^{+\delta}|\}. \quad (B.4)$$

Now by assumption of  $\delta < 1$ , we have

$$d(\mu^*, \mu) < \mu^* \delta + O(\delta). \quad (B.5)$$

because  $e^{\pm\delta} = 1 \pm \delta + O(\delta^2)$  for  $\delta < 1$ .

LEMMA (B.1)

If  $d(\mu^*, \mu) < \mu^* \delta$  and  $\delta < 1$ , then

$$(i) d(e^{-\theta\mu^*}, e^{-\theta\mu}) < \theta e^{-\theta\mu^*} d(\mu^*, \mu),$$

$$(ii) d(e^{-\mu^*}, e^{-\mu}) < e^{-\mu^*} d(\mu^*, \mu),$$

$$(iii) d(\mu^* e^{-\mu^*}, \mu e^{-\mu}) < \mu^* e^{-\mu^*} d(\mu^*, \mu).$$

PROOF : Consider

$$e^{-\theta\mu^*} - e^{-\theta\mu} = e^{-\theta\mu^*}(1 - e^{\theta(\mu^* - \mu)}).$$

If we choose  $\delta$  very small, we have  $e^{\theta(\mu^* - \mu)} \approx 1 + \theta(\mu^* - \mu)$ . Hence

$$e^{-\theta\mu^*} - e^{-\theta\mu} \approx \theta e^{-\mu^*}(\mu - \mu^*) \leq \theta e^{\theta\mu^*} |\mu - \mu^*|.$$

This leads to (i). Similarly (ii) can be shown. Finally, consider

$$\mu^* e^{-\mu^*} - \mu e^{-\mu} = \mu^* e^{-\mu^*} (1 - \frac{\mu}{\mu^*} e^{\mu^* - \mu}).$$

By choosing  $\delta$  very small, we have  $e^{\mu^* - \mu} \approx 1 + \mu^* - \mu$  and  $\frac{\mu}{\mu^*} \approx 1$ . Therefore

$$\mu^* e^{-\mu^*} - \mu e^{-\mu} = \mu^* e^{-\mu^*} (\mu - \mu^*) \leq \mu^* e^{-\mu^*} |\mu^* - \mu|.$$

This is the desired result of (iii).

Now we prove the Proposition using Lemma(B.1).

By (B.2) and (B.3), we have

$$\begin{aligned} S^* - \hat{P}_1 &= \theta(G^* e^{-G^*} - G e^{-G}) \\ &= \theta(e^{G'} - e^{-G'}) + (e^{-\theta G'} - e^{-\theta G}). \end{aligned}$$

After some algebraic manipulation using the fact that  $G = \lambda + \mu$  and  $G^* = \lambda + \mu^*$ , we may obtain

$$\begin{aligned} S^* - \hat{P}_1 &= \theta e^{-\lambda} (\mu^* e^{-\mu^*} - \mu e^{-\mu}) \\ &\quad + (1 + \lambda\theta) e^{-\lambda} (e^{-\mu} - e^{-\mu^*}) + e^{-\theta\lambda} (e^{-\theta\mu^*} - e^{-\theta\mu}). \end{aligned}$$

Hence, from (B.5) and Lemma(B.1) we can show that for very small  $\delta$

$$\begin{aligned} d(S^* - \hat{P}_1) &< \theta e^{-\lambda} d(\mu^* e^{-\mu^*} - \mu e^{-\mu}) \\ &\quad + |1 - \lambda\theta| e^{-\lambda} d(e^{-\mu}, e^{-\mu^*}) + e^{-\theta\lambda} d(e^{-\theta\mu^*}, e^{-\theta\mu}) \\ &\leq (\mu^{*2} e^{-\mu^*} + \mu^* e^{-\mu^*} + \theta \mu^* e^{-\theta\mu^*}) \delta. \end{aligned}$$

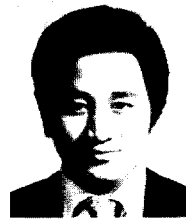
because  $\theta e^{-\lambda} \leq 1$ ,  $|1 - \lambda\theta| e^{-\lambda} \leq 1$  and  $e^{-\theta\lambda} \leq 1$ .

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