論 文

A generalization of Price's theorem with constrained non-Gaussian inputs

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제한적 비가우시안 입력에 대한 Price 정리의 일반화

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Abstract

Price's theorem is generalized for general zero-memory nonlinear functions when inputs are drawn from a sum, called the constrained non-Gaussian, of two or more mutually independent processes of which the first is the Gaussian. An example is given to illustrate the applicability of the generalization.

요 약

Price 정리를 일반적인 제로메모리 함수와 그 입력이 제한적 비가우시안 프로세스에 대하여 일반화 한다. 제한적 비가우시안 프로세스는 둘 이상의 상호 독립인 프로세스들의 합으로 되어 있고, 이 중 첫번째 프로세스는 가우시안인 것을 말한다. 이런 일반화된 정리의 응용성을 보여주기 위해 한 예를 든다.

I. Introduction

Price's theorem[1] has been shown to be useful in evaluating the expected values of the products of the outputs occuring when jointly Gaussian inputs are subjected to zero-memory nonlinear functions. From Ref 1, the statement of the theorem is as follows :

"Assume x_1, x_2, $\cdots$, x_n to be random variables from a Gaussian process whose nth order joint probability density is given by :

$$p(x_1, x_2, \cdots, x_n) = (2\pi)^{-n/2}|M_n|^{1/2} \cdot \exp\left\{-\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{M_{rs}}{|M_n|}(x_r-\bar{x}_r)(x_s-\bar{x}_s)\right\} \tag{1}$$

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where $|M_n|$ is the determinant of $M_n=[\rho_{rs}]$ and $\rho_{rs}=\overline{x_r x_s}-\overline{x}_r\overline{x}_s$ is the correlation of x_r and x_s. The means of x_r and x_s are $\overline{x}_r$ and $\overline{x}_s$, respectively. M_{rs} is the cofactor of ρ_{sr} in M_n.

Let there be n zero-memory nonlinear devices (linearity of course being included as a special case) specified by the input-output relationship $f_i(x)$, $i=1, 2, \cdots, n$. Let each x_i be the signle input to a corresponding $f_i(x)$, and designate the nth-order correlation coefficient of the outputs as :

$$R=\overline{\prod_{i=1}^{n} f_i(x_i)} \tag{2}$$

where the bar denotes the average taken over all x_i. Then, with weak restricitions on the $f_i(x)$, we have the following theorem for the partial derivatives of R with respect to the input correlation coefficients :

$$\frac{\partial^k R}{\prod_{m=1}^{N}(\partial \rho_{r_m s_m})^{k_m}}=\left(\frac{1}{2}\right)^{\sum_{m=1}^{N} k_m \delta_{r_m s_m}} \cdot \overline{\left[\prod_{i=1}^{N} f_i^{\left(\sum_{m=1}^{N}\varepsilon_{im}k_m\right)}(x_i)\right]} \tag{3}$$

where r_m and s_m, $m=1, 2, \cdots, N$, are integers lying between 1 and n, inclusive, and are not necessarily distinct. The k_m are positive integers, with $k=\sum_{m=1}^{N} k_m$, ε_{im} is the number of times i appears in (r_m, s_m). $\delta_{r_m s_m}$ is the Kronecker δ function, $\delta_{r_m s_m}=1$ for $r_m=s_m$, 0 for $r_m \neq s_m$. The symbol $f_i^{(q)}(x_i)$ denotes the qth derivative of $f_i(x)$, taken at x_i.

Furthermore, not only is the above theorem true for inputs having an nth-order joint Gaussian distribution but it holds true *only* for such inputs if the $f_i(x)$ are allowed to be of general form."

In a later correspondence[2], Price noted that the Gaussian random variables in (1) should have been stated to all be of unit variance and his original teorem was extended to the problems involving input processes of a much broader class than the Gaussian. It was stated in Ref. 2 that if the input variables are drawn from a sum of two or more mutually independent random processes of which the first is a process having the set of correlation coefficients $\{\rho_{jk}\}$ $(j, k=1, 2, \cdots, n)$ and unit variance, then the result (3) is true so long as the first process is Gaussian, irrespective of the statistics of the other processes. In particular, when the first is Gaussian, the sum will be called the constrained non-Gaussian process in this paper.

While in Refs. 1 and 2 they consider only separate forms $f_j(x_j)$ $(j=1, 2, \cdots, n)$ for zero-memory nonlinear functions where x_j is j-th input random variable, there are many papers[3-6] in which a general zero-memory nonlinear function $f(x_1, x_2, \cdots, x_n)$ is considered. In Refs. 3 and 4, a proof of the sufficient part of Price's theorem was given for the special case $n=2$. In Ref. 5 a generalized version of Price's theorem was developed by proving its sufficient and necessary (or converse) part for general n. A different method was used in Ref. 6 to obtain the same result as that in Ref. 5. While the developments in Refs. 1-3 appeal to Laplace integral expansions for the nonlinear function, those in Refs. 4 and 5 use less restrictive conditions on the nonlinear function. However, in Refs. 3-6, the derivatives of the expected value of the output were with respect to the correlation coefficients of the whole input process, not with respect to the correlation coefficients of the first process of the constrained non-Gaussian input.

If we want to determine the expected value of the output of the general zero-memeory nonlinear function when the input is Gaussian, then we can use the result of Ref. 5. There is no way to do so, however, when the input is the constrained non-Gaussian. Exceptionally, if the general zero-memeory nonlinear function is of the separate form and the correlation coefficients of the first process of the input are given, we can use the result of Ref. 2. Thus a method is desired that extends the result in Ref. 2 for general zero-memory nonlinear function. When the input or the first process need not have unit variance, it is general to use the set of covariances than correlation coefficients. If we consider a case where the covariances $\lambda_{jk}(j \neq k)$ of the input are equivalent to those of the first process of the constrained non-Gaussian input for a general zero-memory nonlinear function, a generalization of Ref. 2 will be very useful in evaluating the expected value of the output.

In this paper, the result in Ref. 2 is generalized for a general zero-memory nonlinear function when the input is constrained non-Gaussian. Since the jointly Gaussian input can be considered as a special case of the constrained non-Gaussian, the first result in this paper naturally leads to the generalization of Ref. 1 which coincides with the result in Ref. 5 for inputs having unit variances. In addition, the theorem in Ref. 5 is generalized to include inputs with arbitrary variances. In the proofs of these generalizations, Laplace integral expansions are avoided for the general zero-memory nonlinear function by assuming some conditions on the function as in Ref. 4 or 5.

This paper is organized as follows. In Section II, *Theorme 1* is a generalization of the result in Ref. 2 and *Corollary* 1 is a generalization of the result in Ref. 1. *Theorme 2* in Section II can be considered as a generalization of the result Ref. 5. An example of the linear rectifier correlator is shown in Section III to illustrate the applicability of the theorems.

II. Generalization of Price's theorem

We assume that x_j, g_j, and w_j ($j=1, 2, \cdots, n$) are random variables and that $f(x_1, x_2, \cdots, x_n)$ is a general zero-memory nonlinear function and suitably conditioned as in Ref. 4 for n-dimensional case. Let $X=[x_1, x_2, \cdots, x_n]^T$ (T stands for the transpose operation), $G=[g_1, g_2, \cdots, g_n]^T$, and $W=[w_1, w_2, \cdots, w_n]^T$. We denote the joint probability density functions of x_j, g_j, and w_j by $p_X(\cdot)$, $p_G(\cdot)$, and $p_W(\cdot)$, respectively.

Theorem 1 : Let $\lambda_{g,jk}$ be the covariance of g_j and g_k for $j, k=1, 2, \cdots, n$. Then,

$$\frac{\partial E\{f(X)\}}{\partial \lambda_{g,jk}} = \left(\frac{1}{2}\right)^{\delta_{jk}} E\left\{\frac{\partial^2 f(X)}{\partial x_j \partial x_k}\right\} \text{ for } j, k=1, 2, \cdots, n \tag{4}$$

if and only if 1) $x_j = g_j + w_j$, 2) g_j are jointly Gaussian, and 3) g_j and w_k for and j, k are mutually independent, where δ_{jk} denotes the Kronecker delta function and $E\{\cdot\}$ the expectation of $\{\cdot\}$.

Proof. Since g_j are jointly Gaussian, their joint characteristic function Φ_G is

$$\Phi_G(U) = \exp\left(-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_{g,jk}u_j u_k + i\sum_{j=1}^{n}u_j m_{g_j}\right) \tag{5}$$

where $U=[u_1, u_2, \cdots, u_n]^T$, $i=\sqrt{-1}$, and m_{g_j} denotes the mean of g_j. The independence of g_j and w_k yields

$$p_X(X) = p_G(X) * p_W(X) \tag{6a}$$

or

$$\Phi_X(U) = \Phi_G(U)\,\Phi_W(U) \tag{6b}$$

where $*$ denotes the convolution, and Φ_X and Φ_W are the characteristic functions of X and W, respectively. Using (6) and (5), we obtain

$$\begin{aligned}
&\frac{\partial E\{f(X)\}}{\partial \lambda_{g,jk}} \\
&= \frac{\partial}{\partial \lambda_{g,jk}} \int_{R^n} f(X)\, p_X(X)\, dX \\
&= \frac{\partial}{\partial \lambda_{g,jk}} \int_{R^n} f(X)\left[\left(\frac{1}{2\pi}\right)^n \int_{R^n} \Phi_G(U)\Phi_W(U)\exp(-iU^T X)dU\right] dX \\
&= \int_{R^n} f(X)\left[\left(\frac{1}{2\pi}\right)^n \int_{R^n}\left(\frac{1}{2}\right)^{\delta_{jk}}(-u_j u_k)\,\Phi_G(U)\Phi_W(U) \right. \\
&\qquad \left. \cdot \exp(-iU^T X)\, dU\right] dX \\
&= \left(\frac{1}{2}\right)^{\delta_{jk}} \int_{R^n} f(X)\, \frac{\partial^2 p_X(X)}{\partial x_j \partial x_k}\, dX.
\end{aligned} \tag{7}$$

Since

$$\lim_{x_j\to\pm\infty}\frac{p_X(X)}{p_G(X)}=\lim_{x_j\to\pm\infty}\frac{\partial p_X(X)}{\partial x_k}\Big/\frac{\partial p_G(X)}{\partial x_k}=1 \text{ for any } j,\ k \quad (8)$$

from (6a), and

$$\lim_{x_j\to\pm\infty}\frac{\partial f(X)}{\partial x_k}p_G(X)=\lim_{x_j\to\pm\infty}f(X)\frac{\partial p_G(X)}{\partial x_k}=0 \text{ for any } j,\ k \quad (9)$$

under the assumption that $f(X)$ is suitably conditioned as in Ref. 4, we obtain

$$\lim_{x_j\to\pm\infty}\frac{\partial f(X)}{\partial x_k}p_X(X)=\lim_{x_j\to\pm\infty}f(X)\frac{\partial p_X(X)}{\partial x_k}=0 \text{ for any } j,\ k \quad (10)$$

Integrating (7) by parts and using (10), we have

$$\frac{\partial E\{f(X)\}}{\partial\lambda_{g,jk}}=\left(\frac{1}{2}\right)^{\delta_{jk}}\int_{R^n}\frac{\partial^2 f(X)}{\partial x_j\,\partial x_k}p_X(X)\,dX$$

$$=\left(\frac{1}{2}\right)^{\delta_{jk}}E\left\{\frac{\partial^2 f(X)}{\partial x_j\,\partial x_k}\right\}. \quad (11)$$

Let us now prove the necessary part. From the right hand side of (4), we have

$$\left(\frac{1}{2}\right)^{\delta_{jk}}E\left\{\frac{\partial^2 f(X)}{\partial x_j\,\partial x_k}\right\}$$

$$=\left(\frac{1}{2}\right)^{\delta_{jk}}\int_{R^n}f(X)\frac{\partial^2 p_X(X)}{\partial x_j\,\partial x_k}dX$$

$$=\int_{R^n}f(X)\Big[\left(\frac{1}{2\pi}\right)^{-n}\int_{R^n}\left(\frac{1}{2}\right)^{\delta_{jk}}(-u_ju_k)\,\Phi_X(U)$$

$$\cdot\exp(-iU^TX)dU\Big]\,dX. \quad (12)$$

Similarly, the left hand side of (4) becomes

$$\frac{\partial E\{f(X)\}}{\partial\lambda_{g,jk}}$$

$$=\int_{R^n}f(X)\Big[\left(\frac{1}{2\pi}\right)^{-n}\int_{R^n}\frac{\partial\Phi_X(U)}{\partial\lambda_{g,jk}}\exp(-iU^TX)\,dU\Big]dX. \quad (13)$$

Since (4) is an identity for $f(X)$, equating the right hand side of (13) with that of (12) yields

$$\frac{\partial\Phi_X(U)}{\partial\lambda_{g,jk}}=\left(\frac{1}{2}\right)^{\delta_{jk}}(-u_ju_k)\,\Phi_X(U). \quad (14)$$

Integrating (14) for all pairs $(j,\ k)$ and taking into account that $\lambda_{g,jk}=\lambda_{g,kj}$, we have

$$\log\Phi_X(U)=-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_{g,jk}u_ju_k+h_1(U)$$

$$=\log\Phi_G(U)+\Big[-i\sum_{j=1}^{n}u_jm_{g_j}+h_1(U)\Big] \quad (15)$$

where $h_1(U)$ is an arbitrary function independent of $\lambda_{g,jk}$ and $\Phi_G(U)$ the characteristic function of jointly Gaussian random varibles defined in (5). If we let the bracket in (15) be $\log\Phi_W(U)$, (15) implies that $x_j=g_j+w_j$ where g_j and w_k for any $j,\ k$ are mutually independent. This completes the proof of *Theorem 1*.

Theorem 1 is a generalization of the result in Ref. 2, since the function $f(X)$ in (4) clearly includes the functions of the form $f_1(x_1)\,f_2(x_2)\cdots f_n(x_n)$ considered in Ref. 2 as special cases.

Corollary 1: Let $\lambda_{x,jk}$ be the covariance of x_j and x_k for $j,\ k=1,\ 2,\ \cdots,\ n$. Then,

$$\frac{\partial E\{f(X)\}}{\partial\lambda_{x,jk}}=\left(\frac{1}{2}\right)^{\delta_{jk}}E\left\{\frac{\partial^2 f(X)}{\partial x_j\,\partial x_k}\right\} \text{ for } j,\ k=1,\ 2,\ \cdots,\ n \quad (16)$$

if and only if x_j are jonintly Gaussian.

Proof. In *Theorem 1*, if $w_j=0$, then $x_j=g_j(j=1,\ 2,\ \cdots,\ n)$. Thus the proof of the sufficient part of *Corollary* 1 is obvious from that of *Theorem 1*.

Let us prove the necessary part of *Corollary 1*. If we follow the proof of the necessary part of *Theorem 1* with $\lambda_{g,jk}$ replaced by $\lambda_{x,jk}$, then we obtain

$$\log\Phi_X(U)=\Big[-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_{x,jk}u_ju_k\Big]+h_2(U) \quad (17)$$

where $h_2(U)$ is an arbitrary function independent of $\lambda_{x,jk}$. Let y_j and z_j be the random variables associated with the bracket and $h_2(U)$ in (17), respectively. Then it follows that $x_j=y_j+z_j$ where y_j and z_k for any $j,\ k$ are mutually independent. From (5), it is easy to see that y_j are zero-mean jointly Gaussian with covariances $\lambda_{x,jk}$. Since x_j have the covariances $\lambda_{x,jk}$ by assumption, the covariance of z_j and z_k must be zero for all j, k. Therefore z_j should be constants, which are the means of x_j, m_{x_j}. Thus we have

$$h_2(U) = i\sum_{j=1}^{n} u_j m_{x_j} \tag{18}$$

which states, together with (17), that x_j are jointly Gaussian. This completes the proof of *Corollary 1*.

Corollary 1 is a generalization of the result of Ref. 1 and different from that of Ref. 5 in that the covariances $\lambda_{x,jj}$ are involved in (16) while in Ref. 5 only the covariances $\lambda_{x,jk}$ for $j \neq k$ are considered under unit-variance input assumption. Moreover, the following *Theorem 2* tells us that *Corollary 1* also holds true by doing without $(1/2)^{\delta_{jk}}$ in (16), i.e., considering only $\lambda_{x,jk}$ for $j \neq k$.

Theorem 2: Let $\lambda_{x,jk}$ be the covariance of x_j and x_k for $j, k=1, 2, \cdots, n$. Then,

$$\frac{\partial E\{f(X)\}}{\partial \lambda_{x,jk}} = E\left\{\frac{\partial^2 f(X)}{\partial x_j \partial x_k}\right\} \text{ for } j \neq k \tag{19}$$

if and only if x_j are jointly Gaussian.

Proof: The proof of the sufficient part of *Theorem 2* is obvious from that of *Corollary 1*.

Let us consider the necessary part. If we follow the proof of the necessary part of *Theorem 1* with $\lambda_{g,jk}$ replaced by $\lambda_{x,jk}$, we obtain

$$\frac{\partial \Phi_X(U)}{\partial \lambda_{x,jk}} = (-u_j u_k)\,\Phi_X(U) \text{ for } j \neq k, \tag{20}$$

from (14). Integrating (20) for all $j \neq k$ and taking into account that $\lambda_{x,jk} = \lambda_{x,kj}$ give

$$\log \Phi_X(U) = -\sum_{j \neq k} \lambda_{x,jk} u_j u_k + h_3(U) \tag{21}$$

where $h_3(U)$ is an arbitrary function independent of $\lambda_{x,jk}$ for $j \neq k$. From the properties of characteristic function, we obtain

$$\lambda_{x,jj} = -\frac{\partial^2 \Phi_X(U)}{\partial u_j^2}\bigg|_{U=\vec{0}} - m_{x_j}^2 = -\frac{\partial^2 h_3(U)}{\partial u_j^2}\bigg|_{U=\vec{0}} \tag{22}$$

where m_{x_j} denotes the mean of x_j and $\vec{0}$ the nx1 all-zero vector. From (22) it follows that $h_3(U)$ should be of the form

$$h_3(U) = -\frac{1}{2}\sum_{j=1}^{n} \lambda_{x,jj} u_j^2 + h_4(U) \tag{23}$$

where $h_4(U)$ is a function which does not have terms containing u_j^2, $j=1, 2, \cdots, n$. Substituting (23) in (21) yields

$$\log \Phi_X(U) = \left[-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_{x,jk} u_j u_k\right] + h_4(U). \tag{24}$$

Except that $h_4(U)$ may depend on $\lambda_{x,jj}$, (24) is of the same form as (17). Therfore we have $h_4(U) = i\sum_{j=1}^{n} u_j m_{x_j}$ as in (18), which states together with (24) that x_j are jointly Gaussian. This completes the proof of *Theorem 2*.

When the inputs are unit-variance, we have the result of Ref. 5 from *Theorem 2*. Note that the proof of *Theorem 2* is accomplished by using the properties of characteristic function and is simpler than that given in Ref. 5 for unit-variance input case.

Ⅲ. An example

We consider the situation where $n=2$ and assume the followings: 1) $x_j = g_j + w_j (j=1, 2)$, 2) all random variables are zero-mean, 3) g_1 and g_2 are jointly Gaussian with $\lambda_{g,11} = \sigma_{g_1}^2$, $\lambda_{g,22} = \sigma_{g_2}^2$, and $\lambda_{g,12} = \lambda_g$ for notational simplicity, and 4) g_j and w_k (j, $k=1, 2$) are mutually independent. For convenience, we assume that w_1 and w_2 are mutually independent so that $\lambda_{g,12} = \lambda_g$.

Let the probability density functions of w_1 and w_2 be the ε-contaminated (or ε-mixtured) Gaussian[7]:

$$p_{w_1}(w_1)=\varepsilon_{11}N_{w_1}(0,\ \sigma_{w_{11}}^2)+\varepsilon_{12}N_{w_1}(0,\ \sigma_{w_{12}}^2) \tag{25a}$$

and

$$p_{w_2}(w_2)=\varepsilon_{21}N_{w_2}(0,\ \sigma_{w_{21}}^2)+\varepsilon_{22}N_{w_2}(0,\ \sigma_{w_{22}}^2) \tag{25b}$$

where $\varepsilon_{11}=1-\varepsilon_{12}(0\le\varepsilon_{12}\le 1)$, $\varepsilon_{21}=1-\varepsilon_{22}(0\le\varepsilon_{22}\le 1)$, and $N_w(m,\ \sigma^2)$ denotes the Gaussian density with mean m and variance σ^2. Then the joint probability density function of x_1 and x_2 becomes

$$p_{x_1,x_2}(x_1,\ x_2)$$

$$=N_{x_1,x_2}(0,\ \sigma_{g_1}^2,\ \sigma_{g_2}^2,\ \lambda_g)*\left\{\sum_{j=1}^{2}\sum_{k=1}^{2}\varepsilon_{1j}\,\varepsilon_{2k}N_{x_1}(0,\ \sigma_{w_{1j}}^2)N_{x_2}(0,\ \sigma_{w_{2k}}^2)\right\} \tag{26}$$

where $N_{x_1,x_2}(m,\ \sigma_1^2,\ \sigma_2^2,\ \lambda)$ denotes the joint Gaussian density with identical mean m, variance σ_1^2 and σ_2^2, and covariance λ. Since the brace in (26) is separable with respect to x_1 and x_2, (26) becomes

$$p_{x_1,x_2}(x_1,\ x_2)$$

$$=\sum_{j=1}^{2}\sum_{k=1}^{2}\varepsilon_{1j}\,\varepsilon_{2k}\{N_{x_1,x_2}(0,\ \sigma_{g_1}^2,\ \sigma_{g_2}^2\ \lambda_g)*N_{x_1}(0,\ \sigma_{w_{1j}}^2)\}*N_{x_2}(0,\ \sigma_{w_{2k}}^2)$$

$$=\sum_{j=1}^{2}\sum_{k=1}^{2}\varepsilon_{1j}\,\varepsilon_{2k}\,N_{x_1,x_2}(0,\ (\sigma_{g_1}^2+\sigma_{w_{1j}}^1),\ \sigma_{g_2}^2,\ \lambda_g)*N_{x_2}(0,\ \sigma_{w_{2k}}^2)$$

$$=\sum_{j=1}^{2}\sum_{k=1}^{2}\varepsilon_{1j}\,\varepsilon_{2k}\,N_{x_1,x_2}(0,\ (\sigma_{g_1}^2+\sigma_{w_{1j}}^1),\ (\sigma_{g_2}^2+\sigma_{w_{2k}}^2),\ \lambda_g). \tag{27}$$

Now, we consider the linear-rectifier correlator[3], i.e.,

$$f(x_1,\ x_2)=|x_1+x_2|-|x_1-x_1| \tag{28}$$

and use *Theorem 1* to evaluate $E\{f(x_1,\ x_2)\}$. It is easily shown that

$$\frac{\partial^2 f(x_1,\ x_2)}{\partial x_1\,\partial x_2}=2\{\delta(x_1+x_2)+\delta(x_1-x_2)\}. \tag{29}$$

Using *Theorem 1* togeher with (27) and (29), and following straightforward caculations yield

$$\frac{\partial E\{f(x_1,\ x_2)\}}{\partial \lambda_g}=\frac{1}{\sqrt{\pi}}\sum_{j=1}^{2}\sum_{k=1}^{2}\varepsilon_{1j}\,\varepsilon_{2k}$$

$$\cdot\left(\frac{1}{\sqrt{b_{jk}+\lambda_g}}+\frac{1}{\sqrt{b_{jk}-\lambda_g}}\right) \tag{30}$$

where $b_{jk}=(\sigma_{g_1}^2+\sigma_{g_2}^2+\sigma_{w_{1j}}^2+\sigma_{w_{2k}}^2)/2$. Integrating (30) with respect to λ_g, we have

$$E\{f(x_1,\ x_2)\}=\frac{1}{\sqrt{\pi}}\sum_{j=1}^{2}\sum_{k=1}^{2}\varepsilon_{1j}\,\varepsilon_{2k}$$

$$\cdot\left(\sqrt{b_{jk}+\lambda_g}-\sqrt{b_{jk}-\lambda_g}\right), \tag{31}$$

noting that the integration constant becomes zero because $E\{|x_1-x_2|\}=E\{|x_1+x_2|\}$ when $\lambda_g=0$. As a special when $\varepsilon_{12}=\varepsilon_{22}=0$, $\sigma_{w_{jk}}^2=0(j,\ k=1,\ 2)$, and $\sigma_{g_1}^2=\sigma_{g_2}^2=1$, the result (31) becomes that in Ref. 3.

Ⅳ. Conclusions

In this paper generalizations of Price's theorem were considered for various environment. First, a generalization is considered for constrained non-Gaussian inputs and general zero-memeory nonlinear functions. A special case of the generalization was shown to be useful for Gaussian inputs with arbitrary variance and general zero-memory nonlinear functions. Second, a modification of the generalization was considered for Gaussian inputs with arbitrary variance and general zero-memory functions.

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