Asymptotic Characteristics of MSE-Optimal Scalar Quantizers for Generalized Gamma Sources

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ABSTRACT

Characteristics, such as the support limit and distortions, of minimum mean-squared error (MSE) N-level uniform and nonuniform scalar quantizers are studied for the family of the generalized gamma density functions as N increases. For the study, MSE-optimal scalar quantizers are designed at integer rates from 1 to 16 bits/sample, and their characteristics are compared with corresponding asymptotic formulas. The results show that the support limit formulas are generally accurate. They also show that the distortion of nonuniform quantizers is observed to converge to the Panter-Dite asymptotic constant, whereas the distortion of uniform quantizers exhibits slow or even stagnant convergence to its corresponding Hui-Neuhoff asymptotic constant at the studied rate range, though it may stay at a close proximity to the asymptotic constant for the Rayleigh and Laplacian pdfs. Additional terms in the asymptote result in quite considerable accuracy improvement, making the formulas useful especially when rate is 8 or greater.

Key words : asymptotic, optimal, scalar, quantizer, generalized gamma

Ⅰ. Introduction

The goal of the paper is to study asymptotic characteristics, specifically the mean-squared error (MSE) distortion, the support limit and the innermost threshold, of MSE-optimal N-level scalar quantizers as N increases. Toward this goal, we design MSE-optimal uniform and nonuniform scalar quantizers at integer rate $R \equiv \log_2 N$ from 1 to 16 for the common generalized gamma probability density functions (pdfs) [1] and compare their characteristics with the corresponding formulas in order to gain insights and assess accuracy of the formulas in the studied rate range.

The results show (1) that the asymptotic formulas for the support limit and the innermost threshold are generally accurate in both uniform and nonuniform quantizers for all the considered pdfs, (2) that the distortion of nonuniform quantizers is observed to converge in such a consistent manner that it can be predicted quite accurately from the Panter-Dite asymptotic constant when $R \geq 8$, and (3) that the distortion of uniform quantizers exhibits slow convergence to its corresponding Hui-Neuhoff asymptotic constant: from a close proximity to the limit for the Rayleigh and Laplacian; and from a distance for the Gaussian and gamma pdfs. Additional terms in the asymptote formulas improve accuracy quite considerably and in many cases dramatically, making the formulas useful especially when $R \geq 8$.

The numerical data obtained for the common generalized gamma pdfs in Table 1 for the study are generally consistent with those previously reported [2]~[7], whose considered rates range to 6 or in some cases 8. In this paper we extend the study numerically to higher rates up to 16 for both uniform and nonuniform quantizers and, more importantly,

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also analytically by providing asymptotic formulas.

The main contribution of the paper is quantification of the accuracy of asymptotic formulas for optimal uniform and nonuniform scalar quantizers, some of which are rederived with more terms than previously given, e.g., [2], [8]. The significance of the paper comes from the fact that the formulas herein enable one to evaluate accurately the characteristics of optimum quantizers without actually designing and/or implementing them. For example, one does not have to design an optimum 300-level uniform/nonuniform quantizer for a Gaussian pdf but can assess its important characteristics including the distortion using the obtained formulas in this paper. To the best of the authors' knowledge the results presented in this paper have not been reported previously in the literature.

The paper begins in Section 2 with the source densities of interest, background on scalar quantization and MSE distortion. Section 3 identifies quantizer characteristics chosen for the study, summarizes from [2], [8] the results on the optimal support limit and asymptotic expressions for distortion of uniform scalar quantizers, finds their counterparts of nonuniform scalar quantizers, and compares them with those of designed quantizers. Finally, Section 5 concludes.

II. Sources and Quantizers

We focus on quantization of an absolutely continuous, real-valued random variable X with a pdf denoted $p(x)$. As in [2], we consider only symmetric pdfs because this simplifies the analysis and because many common pdfs are symmetric, e.g., Gaussian. We assume throughout that X has finite variance σ^2.

2.1. Source Density Functions

We focus on the generalized gamma pdfs. A generalized gamma (G-Γ) density [1] takes the form

$$p(x) = \mu|x|^{\beta}\exp(-\lambda|x|^{\alpha})$$

where $\alpha > 0,\ \beta > -1,\ \lambda > 0,$

$$\mu = \frac{\alpha}{2}\frac{\lambda^{(\beta+1)/\alpha}}{\Gamma\left(\frac{\beta+1}{\alpha}\right)},\ \sigma^2 = \mathrm{var}(X) = \frac{1}{\lambda^{2/\alpha}}\frac{\Gamma\left(\frac{\beta+3}{\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\alpha}\right)},$$

and $\Gamma(z) = \int_0^{\infty} x^{a-1}e^{-x}dx$ denotes the gamma function. It is parameterized by (α,β,λ) and is symmetric, infinitely differentiable in $(-\infty,\infty)$ except possibly at 0, and hence have a second derivative that is continuous except possibly at 0.

This pdf family includes a broad range of distributions. For example, it includes Gaussian, two-sided Rayleigh, Laplacian and gamma pdfs. We will call these the common generalized gamma pdfs, whose parameters are given in Table 1, whose last two columns C_{HN} and C_{PD} are explained later in Section 3. The unit variance requires that $\lambda = \left\{\Gamma\left(\frac{\beta+3}{\alpha}\right)/\Gamma\left(\frac{\beta+1}{\alpha}\right)\right\}^{\alpha/2}$.

Table 1: Parameters of common G-Γ pdfs with variance σ^2

	α	β	λ	μ	C_{HN}	C_{PD}
Gaussian	2	0	$\frac{1}{2\sigma^2}$	$\frac{1}{\sigma\sqrt{2\pi}}$	$\frac{4}{3}\sigma^2$	$\frac{\sqrt{3}\pi}{2}\sigma^2$
two-sided Rayleigh	2	1	$\frac{1}{\sigma^2}$	$\frac{1}{\sigma^2}$	$\frac{2}{3}\sigma^2$	$\frac{3}{4}\Gamma^3\left(\frac{2}{3}\right)\sigma^2$
Laplacian	1	0	$\frac{\sqrt{2}}{\sigma}$	$\frac{1}{\sigma\sqrt{2}}$	$\frac{2}{3}\sigma^2$	$\frac{9}{2}\sigma^2$
gamma	1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2\sigma}$	$\frac{\sqrt[4]{3}}{\sqrt{8\pi\sigma}}$	$\frac{16}{9}\sigma^2$	$\frac{4\sqrt{3}}{\sqrt{\pi}}\Gamma^3\left(\frac{5}{6}\right)\sigma^2$

2.2. Scalar Quantizers

For the previously described pdfs, we focus on uniform and nonuniform scalar quantizers that are symmetric with an even number of levels because they are easier to work with and because for large N these assumptions have negligible effect. That is, the asymptotic expressions we find hold equally for asymmetric quantizers and for quantizers with odd numbers of levels, with the exception of the innermost threshold x_2, defined shortly, because it is specialized for even N.

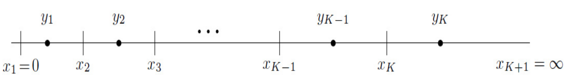

Fig. 1. A symmetric scalar quantizer Q_N with $N = 2K$ levels: the nonnegative half.

As depicted in Figure 1, an N-level symmetric scalar quantizer with an even number $N=2K$ of levels is characterized by its nonnegative thresholds $x_1<x_2<\cdots<x_{K+1}$, with $x_1=0$ and $x_{K+1}=\infty$, and its positive quantization levels $y_1<y_2<\cdots<y_K$. The negative thresholds and levels of Q_N are the reflection of the positive ones. The quantizer will be denoted by its quantization rule Q_N, defined for positive x by $Q_N(x)=y_i$, when $x_i<x<x_{i+1}$, and similarly for negative x. Note that whenever N is mentioned, it will be presumed that $K=N/2$, and likewise whenever K is mentioned, it will be presumed that $N=2K$. Thus, some formulas will be expressed in terms of N, some in terms of K, and some in terms of both.

An N-level symmetric uniform scalar quantizer Q_N^U with an even number of levels and step size Δ is an N-level symmetric scalar quantizer Q_N whose nonnegative half has the following properties: $x_i=(i-1)\Delta$ for $i=1,\cdots,K$; and $y_i=\left(i-\frac{1}{2}\right)\Delta$ for $i=1,\cdots,K$.

We will call x_K the support limit of Q_N, interval $[-x_K,x_K]$ the inner region, and its complement the outer region. Although the granular distortion region $[-(y_K+(y_K-x_K)),y_K+(y_K-x_K)]$ and the overload region, its complement, are commonly used, we prefer to use the inner and the outer regions associated with the support limit x_K, since this simplifies expressions somewhat. We note that the difference is insignificant for large K in the case of the optimal uniform quantizers considered in this study because their step sizes $\Delta\to 0$ as $K\to\infty$.

When applied to a symmetric $p(x)$, the MSE distortion of a symmetric scalar quantizer Q_N with an even number of levels $N=2K$ is given by

$$D(Q_N)=\int_{-\infty}^{\infty}(x-Q_N(x))^2p(x)dx \tag{1}$$
$$=2\sum_{i=1}^{K}\int_{x_i}^{x_{i+1}}(x-y_i)^2p(x)dx.$$

Quantizer Q_N is said to be optimal if it minimizes the distortion. We will express $D(Q_N)$ as the sum of the inner distortion, denoted $D_i(Q_N)$, and the outer distortion, denoted $D_o(Q_N)$, which are given by

$$D_i(Q_N)=\int_{-x_K}^{x_K}(x-Q_N(x))^2p(x)dx \tag{2}$$
$$=2\sum_{i=1}^{K-1}\int_{x_i}^{x_{i+1}}(x-y_i)^2p(x)dx$$

$$D_o(Q_N)=2\int_{x_K}^{\infty}(x-y_K)^2p(x)dx \tag{3}$$

These will be used without the argument, i.e., D, D_i and D_o, when the quantizer under consideration is understood or confusion is not likely. We note that D_i roughly corresponds to the granular distortion when K is large and (y_K-x_K) is finite, which is the case for optimal uniform and nonuniform quantizers for G-Γ pdfs with $\alpha\ge 1$, and similarly D_o to the overload distortion.

III. Characteristics of Optimal Quantizers

In this section specific formulas are given for the characteristics of optimal quantizers and compared with the numerical values of the designed quantizers. Although there may be many different notions of important characteristics of optimal quantizers, we choose, as such, the support limit x_K and distortions D_i, D_o and D. We also include the first positive threshold x_2, which will be called the innermost step size, a characteristic that tells how a quantizer treats small input values. In the case of uniform quantizers, the innermost step size x_2 equals the step size Δ and is related with x_K through $x_2=\Delta=x_K/(K-1)$, whereas that of a nonuniform quantizer has no known general relationship, hence its choice.

These formulas and many others in this paper use the following "o" and "O" notation: $o_N(1)$ denotes a quantity that vanishes as $N\to\infty$; given a function h, $o_N(h(N))$ denotes a quantity such that $o_N(h(N))/h(N)$ vanishes as $N\to\infty$, and $o_\Delta(h(\Delta))$ denotes a quantity such that $o_\Delta(h(\Delta))/h(\Delta)$ vanishes as $\Delta\to 0$. The subscript on "o" will sometimes be dropped when it is clear from context. In addition, given a function or constant h, $O(h(N))$ denotes a quantity such that $O(h(N))/h(N)$ remains bounded as N varies, and $O(h(\Delta))$ is defined similarly.

3.1. Optimal Uniform Quantizers

3.1.1. The Support Limit and the Innermost Step Size

Hui and Neuhoff [2] developed a method for characterizing the asymptotic form of the support limit x_K of an optimal uniform quantizer Q_N^U in terms of N and the density p, and gave specific formulas for certain densities. The step size Δ of an optimal quantizer can of course be found from $\Delta = x_K/(K-1)$.

The optimal Δ can be found setting the derivative of (1) with respect to Δ to zero and solving it or from [2, Thm. 5] or [8, Lemma 20]:

$$\frac{\Delta}{6}(1+o_N(1)) = \frac{4\mu K}{\alpha^2\lambda^2} e^{-\lambda x_K^\alpha} x_K^{\beta-2\alpha+2}(1+o_N(1)) \qquad (4)$$

where $x_K = (K-1)\Delta$. Noting that $\Delta = x_K(K-1) = x_K(1+o_K(1))/K = x_K(1+o_N(1))/K$, then dropping the o terms, and taking logarithms yield

$$\lambda x_K^\alpha = \ln\left(\frac{3N^2}{\alpha\Gamma\left(\frac{\beta+1}{\alpha}\right)}\right) - \frac{2\alpha-\beta-1}{\alpha}\ln(\lambda x_K^\alpha), \qquad (5)$$

which can be put into the following form:

$$t = u - s\ln t \qquad (6)$$

where $t \equiv \lambda x_K^\alpha$, $u \equiv \ln\left(3N^2/\alpha\Gamma\left(\frac{\beta+1}{\alpha}\right)\right)$ and $s \equiv (2\alpha-\beta-1)/\alpha$. An analytic solution to (6) is found in an iterative manner, as indicated in [11, pp. 11~13]: one first starts with a rough solution usually involving the most significant term and then makes refinements thereafter. The procedure is illustrated in Figure 2. First we note that the solution to (6) is the intersection of two functions $f(t) = t$ and $g(t) = u - s\ln t$ for a given u. The initial solution starts with solving $f(t) = u$ for t, that is $t = u$, designated as point A in the figure. A refinement is obtained by evaluating $g(t) = u - s\ln t$ at the initial solution $t = u$, which is $g(u) = u - s\ln u$, point B. When $g(u) = u - s\ln u$ is equated with $f(t) = t$, its solution for t is denoted u_1, giving point C. Next refinement begins with evaluating $g(t)$ at $t = u_1$ at point D and goes on to point E and so on. When proceeded, it yields

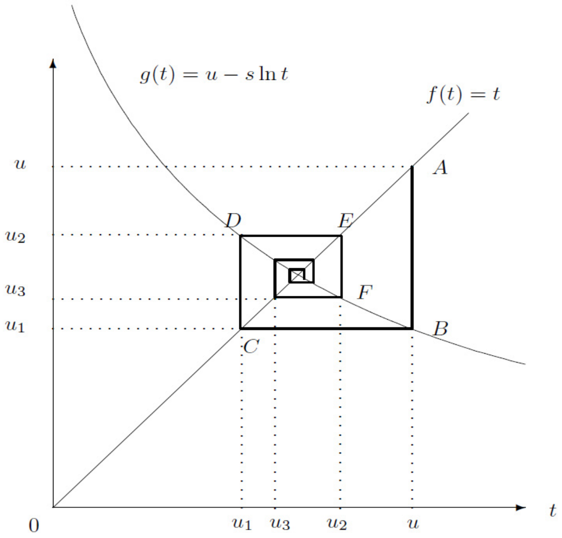

Fig. 2. Illustration of finding an analytic solution to $t = u - s\ln t$

$$t = u - s\ln u + s^2\frac{\ln u}{u} + \frac{s^3}{2}\frac{(\ln u)^2}{u^2} - s^3\frac{\ln u}{u^2} + \frac{s^4}{3}\frac{(\ln u)^3}{u^3} - \frac{3s^4}{2}\frac{(\ln u)^2}{u^3} + s^4\frac{\ln u}{u^3} + o\left(\frac{1}{u^3}\right). \qquad (7)$$

When back substituted and expressed in terms of N, (7) gives the following expression for the support limit x_K:

$$x_K^\alpha = \frac{2\ln N}{\lambda}\left\{1 + c_1\frac{\ln\ln N}{\ln N} + c_2\frac{1}{\ln N} + c_1^2\frac{\ln\ln N}{(\ln N)^2} + c_1c_2\frac{1}{(\ln N)^2} + O\left(\frac{(\ln\ln N)^2}{(\ln N)^3}\right)\right\}, \qquad (8)$$

where

$$c_1 = -\frac{2\alpha-\beta-1}{2\alpha},$$

$$c_2 = -\frac{1}{2}\ln\left(\frac{2^{(2\alpha-\beta-1)/\alpha}\alpha\Gamma\left(\frac{\beta+1}{\alpha}\right)}{3}\right)$$

from which the following expression is derived using the binomial series of

$(1+v)^r = 1 + rv + \frac{r(r-1)}{2!}v^2 + \cdots$ for $|v| < 1$:

$$x_K = \left(\frac{2\ln N}{\lambda}\right)^{1/\alpha}\left\{1 + \frac{c_1}{\alpha}\frac{\ln\ln N}{\ln N} + \frac{c_2}{\alpha}\frac{1}{\ln N} + \frac{(1-\alpha)c_1^2}{2\alpha^2}\frac{(\ln\ln N)^2}{(\ln N)^2} + \left(\frac{c_1^2}{\alpha} + \frac{(1-\alpha)c_1c_2}{\alpha^2}\right)\frac{\ln\ln N}{(\ln N)^2} + \left(\frac{c_1c_2}{\alpha} + \frac{(1-\alpha)c_2^2}{2\alpha^2}\right)\frac{1}{(\ln N)^2} + O\left(\frac{(\ln\ln N)^3}{(\ln N)^3}\right)\right\}. \qquad (9)$$

Eq. (8) is a rederived version of Eq. (48) of [2] with more terms. Then, an expression for the innermost step size x_2 can be found from the fact that $x_K=(K-1)\Delta=\frac{N}{2}\left(1-\frac{2}{N}\right)\Delta$ and hence $x_2=\Delta=\frac{2x_K}{N}\left(1+O\left(\frac{1}{N}\right)\right)$. The result is given in Table 2 as x_2^U.

3.1.2 The Distortions

The following from Thm. 4 of [8] is for the distortion of (not necessarily optimal) uniform quantizers with $x_2=\Delta$ and $x_K=(K-1)\Delta$: for a symmetric uniform quantizer Q_N^U with even $N=2K$ ($K\geq 2$) and step size Δ applied to a symmetric pdf $p(x)$

$$D(Q_N^U)=D_i(Q_N^U)+D_o(Q_N^U) \tag{10}$$

where

$$D_i(Q_N^U)=\frac{\Delta^2}{12}+\int_0^{x_2}\left(2u^2-2u\Delta+\frac{\Delta^2}{3}\right)p(u)du+R_d(\Delta,K) \tag{11}$$

$$D_o(Q_N^U)=2\int_{x_K}^{\infty}u^2p(u)du-4\left(x_K+\frac{\Delta}{2}\right)\int_{x_K}^{\infty}up(u)du+2\left(x_K^2+x_K\Delta+\frac{\Delta^2}{6}\right)\int_{x_K}^{\infty}p(u)du, \tag{12}$$

where $|R_d(\Delta,K)|\leq\frac{(2-2^{-3})\Delta^4}{180}\int_{x_2}^{x_K}|p^{(2)}(u)|du$. From these follow a set of specialized expressions (c.f., Thm. 7 of [8]):

$$D_i(Q_N^U)=\frac{\Delta^2}{12}(1+O(\Delta^{\min(2,\beta+1)})), \tag{13}$$

$$D_o(Q_N^U)=\frac{4\mu}{\alpha^3\lambda^3}e^{-\lambda x_K^\alpha}x_K^{\beta-3\alpha+3}(1+o_N(1)). \tag{14}$$

We note that a slightly sharper result for the Gaussian and Laplacian pdfs ($\beta=0$) replaces the term $O(\Delta^{\min(2,\beta+1)})=O(\Delta)$ in $D_i(Q_N^U)$ with $O(\Delta^2)$. Then for D_i a version of Eq. (33) of [8] is rederived using (8) to include additional terms:

$$D_i(Q_N^U)=\frac{1}{3}\left(\frac{2}{\lambda}\right)^{2/\alpha}\frac{(\ln N)^{2/\alpha}}{N^2}\left\{1+\frac{2c_1}{\alpha}\frac{\ln\ln N}{\ln N}+\frac{2c_2}{\alpha}\frac{1}{\ln N}+\frac{(2-\alpha)c_1^2}{\alpha^2}\frac{(\ln\ln N)^2}{(\ln N)^2}+\left(\frac{2c_1^2}{\alpha}+\frac{2(2-\alpha)c_1c_2}{\alpha^2}\right)\frac{\ln\ln N}{(\ln N)^2}+\left(\frac{2c_1c_2}{\alpha}+\frac{(2-\alpha)c_2^2}{\alpha^2}\right)\frac{1}{(\ln N)^2}+o\left(\frac{1}{(\ln N)^2}\right)\right\} \tag{15}$$

The ratio $D_o(Q_N^U)/D_i(Q_N^U)$ is found from Eq. (43) of [8] to be

$$\frac{D_o(Q_N^U)}{D_i(Q_N^U)}=\frac{c_3}{\ln N}(1+o_N(1)) \tag{16}$$

where $c_3\equiv 1/\alpha$. We note that logarithmic dominance of D_i over D_o in uniform quantization is weaker than linear dominance in nonuniform quantization, as commented following (29).

Then, since $D=D_i+D_o=D_i(1+D_o/D_i)$, we have

$$\frac{N^2D(Q_N^U)}{(\ln N)^{2/\alpha}}=\frac{1}{3}\left(\frac{2}{\lambda}\right)^{2/\alpha}\left\{1+\frac{2c_1}{\alpha}\frac{\ln\ln N}{\ln N}+\left(\frac{2c_2}{\alpha}+c_3\right)\frac{1}{\ln N}+\frac{(2-\alpha)c_1^2}{\alpha^2}\frac{(\ln\ln N)^2}{(\ln N)^2}+\left(\frac{2c_1^2+2c_1c_3}{\alpha}+\frac{2(2-\alpha)c_1c_2}{\alpha^2}\right)\frac{\ln\ln N}{(\ln N)^2}+\left(\frac{2c_1c_2+2c_2c_3}{\alpha}+\frac{(2-\alpha)c_2^2}{\alpha^2}\right)\frac{1}{(\ln N)^2}+o\left(\frac{1}{(\ln N)^2}\right)\right\} \tag{17}$$

We note that the contribution of D_o first appears only in the third term in the bracket, the effect of D_o being logarithmically dominated.

The common G-Γ pdfs have the α value of either 1 or 2. In such cases, (17) reduces for the Gaussian and Rayleigh pdfs ($\alpha=2$ and $c_3=1/2$) to

$$\frac{N^2D(Q_N^U)}{\ln N}=\frac{2}{3\lambda}\left\{1+c_1\frac{\ln\ln N}{\ln N}+(c_2+c_3)\frac{1}{\ln N}+(c_1^2+c_1c_3)\frac{\ln\ln N}{(\ln N)^2}+(c_1c_2+c_2c_3)\frac{1}{(\ln N)^2}+o\left(\frac{1}{(\ln N)^2}\right)\right\} \tag{18}$$

and for the Laplacian and gamma pdfs ($\alpha=1$ and $c_3=1$) to

$$\frac{N^2D(Q_N^U)}{(\ln N)^2}=\frac{4}{3\lambda^2}\left\{1+2c_1\frac{\ln\ln N}{\ln N}+(2c_2+c_3)\frac{1}{\ln N}+c_1^2\frac{(\ln\ln N)^2}{(\ln N)^2}+(2c_1^2+2c_1c_2+2c_1c_3)\frac{\ln\ln N}{(\ln N)^2}+(2c_1c_2+c_2^2+2c_2c_3)\frac{1}{(\ln N)^2}+o\left(\frac{1}{(\ln N)^2}\right)\right\} \tag{19}$$

The asymptotic constant of the distortion is obtained from (17):

$$\lim_{N\to\infty}\frac{N^2D(Q_N^U)}{(\ln N)^{2/\alpha}}=C_{HN}=\frac{1}{3}\left(\frac{2}{\lambda}\right)^{2/\alpha}=\frac{2^{2/\alpha}}{3}\frac{\Gamma\left(\frac{\beta+1}{\alpha}\right)}{\Gamma\left(\frac{\beta+3}{\alpha}\right)}\sigma^2. \tag{20}$$

The limit value in the right hand side will be called the Hui-Neuhoff asymptotic constant, denoted C_{HN}, as it was first reported by them [2]. Table 1 lists C_{HN} for the common G-Γ pdfs.

3.2 Optimal Nonuniform Quantizers

3.2.1 The Support Limit and the Innermost Step Size

For a symmetric quantizer Q_N^* soptimized for a G-Γ density parameterized by (α,β,λ), the support limit x_K is given by the following [9]

$$x_K^\alpha = \frac{3}{\lambda}\ln N - \frac{3\alpha-\beta-3}{\alpha\lambda}\ln\ln N - \frac{3}{\lambda}\ln\left(3\Gamma\left(\frac{\beta+3}{3\alpha}\right)\right) + o_N(1) \quad (21)$$

An expression for the innermost step size x_2 can be heuristically found in terms of *N* by applying, e.g., the differentiation method [9]: solve $\frac{dD(Q_N)}{dx_2}=0$ for x_2, where $D(Q_N)$ in (1), in order to show the dependence on x_2 explicitly, is rewritten as

$$D(Q_N) = 2\left\{\int_0^{x_2}(x-y_1)^2 p(x)dx + \int_{x_2}^{\infty}(x-Q_N(x))^2 p(x)dx\right\}$$

where y_1 is the optimal quantization level of $[0,x_2]$, i.e., its centroid with respect to *p*. When the second integral term is approximated as a Panter-Dite formula-like expression [10] for optimal Q_N, we have

$$D(Q_N) \approx 2\int_0^{x_2}(x-y_1)^2 p(x)dx + \frac{1}{12N^2}\left(2\int_{x_2}^{\infty}p^{1/3}(x)dx\right)^3 \quad (22)$$

The use of the derivative of the right hand side of (22) as that of the original $D(Q_N)$ makes the approach heuristic. When proceeded, x_2 is found solving

$$(x_2-y_1)p^{1/3}(x_2) - \frac{1}{N}\int_{x_2}^{\infty}p^{1/3}(x)dx = 0. \quad (23)$$

Solving (23), we obtain

$$x_2 = \left(\frac{2}{N}\right)^{3/(\beta+3)}\left(\frac{3}{\lambda}\right)^{1/\alpha}\left(\frac{\beta+2}{2\alpha}\Gamma\left(\frac{\beta+3}{3\alpha}\right)\right)^{3/(\beta+3)}\left\{1+O\left(\frac{1}{N^{\min(1,3\alpha/(\beta+3))}}\right)\right\}. \quad (24)$$

We note that the primary dependence of x_2 on *N* is of $1/N^{3/(\beta+3)}$ with β being the only affecting factor and it makes sense because $p(x)=\mu|x|^\beta\exp(-\lambda|x|^\alpha)\approx\mu|x|^\beta$ around $x\approx 0$. We have specifically $x_2 = \frac{\sigma\sqrt{6\pi}}{N}\left\{1+O\left(\frac{1}{N}\right)\right\}$ for the Gaussian pdf; $\frac{\sigma\sqrt{3}}{N^{3/4}}\left(\frac{3}{2}\Gamma\left(\frac{2}{3}\right)\right)^{3/4}\left\{1+O\left(\frac{1}{N}\right)\right\}$ for the two-sided Rayleigh; $\frac{3\sigma\sqrt{2}}{N}\left\{1+O\left(\frac{1}{N}\right)\right\}$ for the Laplacian; and $\frac{2\sigma\sqrt{3}}{N^{6/5}}\left(\frac{3}{2}\Gamma\left(\frac{5}{6}\right)\right)^{6/5}\left\{1+O\left(\frac{1}{N}\right)\right\}$ for the gamma.

3.2.2 The Distortions

The asymptotic formula for the inner distortion D_i is provided by the Panter-Dite formula [10]

$$D_i(Q_N) = \frac{1}{12N^2}\left(2\int_0^{x_K}p^{1/3}(x)dx\right)^3\{1+o_N(1)\},$$

which are evaluated to yield

$$D_i(Q_N^*) = \frac{3^{(\beta-\alpha+3)/\alpha}}{\alpha^2\lambda^{2/\alpha}}\frac{\Gamma^3\left(\frac{\beta+3}{3\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\alpha}\right)}\frac{1}{N^2}\{1+o_N(1)\}, \quad (25)$$

which reduces to $\frac{\sqrt{3}\pi}{2}\frac{\sigma^2}{N^2}\{1+o_N(1)\}$ for the Gaussian pdf and $\frac{9}{2}\frac{\sigma^2}{N^2}\{1+o_N(1)\}$ for the Laplacian. These agree well with the previous reports, e.g., [6]. An asymptotic expression for the outer distortion $D_o(Q_N)$ in (3) can be obtained using the formula for optimal x_K and noting that y_K is the centroid of $[x_K,\infty)$ with respect to *p*, i.e.,

$$y_K = \frac{\int_{x_K}^{\infty}xp(x)dx}{\int_{x_K}^{\infty}p(x)dx} = \frac{\Gamma\left(\frac{\beta+2}{\alpha},\lambda x_K^\alpha\right)}{\lambda^{1/\alpha}\Gamma\left(\frac{\beta+1}{\alpha},\lambda x_K^\alpha\right)} \quad (26)$$

where $\Gamma(a,z)=\int_z^{\infty}x^{a-1}e^{-x}dx$ is the complementary incomplete gamma function. When using the asymptotic expansion [11, p.66] of

$$\Gamma(a,z) = e^{-z}z^{a-1}\left\{1+\frac{a-1}{z}+\frac{(a-1)(a-2)}{z^2}+O(z^{-3})\right\} \quad (27)$$

as $z\to\infty$, we obtain

$$y_K = x_K\left\{1+\frac{1}{\alpha\lambda}\frac{1}{x_K^\alpha}+\frac{\beta-2\alpha+2}{\alpha^2\lambda^2}\frac{1}{x_K^{2\alpha}}+O\left(\frac{1}{x_K^{3\alpha}}\right)\right\}. \quad (28)$$

Then the outer distortion D_o is derived from (3)

$$D_o(Q_N^*) = \frac{2\mu}{\alpha^3\lambda^3}x_K^{\beta-3\alpha+3}e^{-\lambda x_K^\alpha}\left\{1+O\left(\frac{1}{x_K^{\min(1,\alpha)}}\right)\right\},$$

which yields

$$D_o(Q_N^*) = \frac{3^{(\beta+3)/\alpha}}{\alpha^2\lambda^{2/\alpha}} \frac{\Gamma^3\left(\frac{\beta+3}{\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\alpha}\right)} \frac{1}{N^3}\{1+o_N(1)\} \qquad (29)$$

with substitution (21) for x_K. As is well known, D_o is in the order of $1/N^3$ and, for example, $D_o = \frac{3\pi\sqrt{3}}{2}\frac{\sigma^2}{N^2}\{1+o_N(1)\}$ for the Gaussian pdf and $D_o = \frac{27}{2}\frac{\sigma^2}{N^3}\{1+o_N(1)\}$ for the Laplacian. In the case of G-Γ pdfs, the ratio $D_o(Q_N^*)/D_i(Q_N^*) = \frac{3}{N}\{1+o_N(1)\}$ shows strong dominance of $D_i(Q_N^*)$, which contrasts to that of the uniform case $D_o(Q_N^U)/D_i(Q_N^U) = \frac{1}{\alpha}\frac{1}{\ln N}\{1+o_N(1)\}$ in (16).

The distortion $D(Q_N^*)$ has the following limit value:

$$\lim_{N\to\infty} N^2 D(Q_N^*) = C_{PD} = \frac{3^{(\beta-\alpha+3)/\alpha}}{\alpha^2\lambda^{2/\alpha}} \frac{\Gamma^3\left(\frac{\beta+3}{3\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\alpha}\right)}$$

$$= \frac{3^{(\beta-\alpha+3)/\alpha}}{\alpha^2} \frac{\Gamma^3\left(\frac{\beta+3}{3\alpha}\right)}{\Gamma\left(\frac{\beta+3}{\alpha}\right)}\sigma^2. \qquad (30)$$

We refer to the limit as the Panter-Dite asymptotic constant. For the common G-Γ pdfs it is tabulated in Table 1.

IV. Numerical Results and Discussion

Tables 3~6 show the innermost step size x_2, support limit x_K, and distortion D of optimal symmetric uniform and nonuniform quantizers for the common G-Γ pdfs with the unit variance. The superscript U denotes those of uniform quantizers and similarly * nonuniform quantizers. These optimal quantizers are designed using Max's approach [3] and agree generally with the previous reports as already mentioned in Section 1.

Figure 3 shows the values of x_2 and x_K for optimal uniform and nonuniform quantizers for the common unit-variance G-Γ pdfs. Their corresponding formulas that are used to obtain numerical results are summarized in Table 2. In both uniform and nonuniform quantizers the formulas are generally accurate in that they predict the true values approximately within 2% of error or smaller at $R \geq 6$ for the Gaussian (G) and two-sided Rayleigh (R); $R \geq 7$ for the Laplacian (L); and $R \geq 8$ for the gamma (Γ) pdf, and their accuracies improve with the rate, making each pair of the true values and the corresponding formula appear to overlap as R increases. It is interesting to observe that $x_K^U = \frac{4\sqrt{3}}{3}\ln N(1+o_N(1)) = 2.31\ln N(1+o_N(1))$ for Γ and $x_K^* = \frac{3}{\sqrt{2}}\ln N(1+o_N(1)) = 2.12\ln N(1+o_N(1))$ for L stay close to each other at the studied rate range.

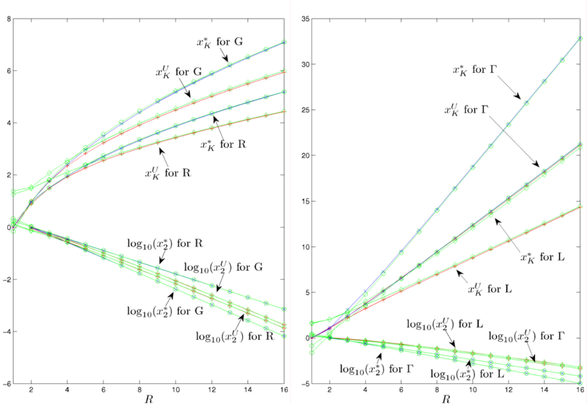

Fig. 3. Thresholds x_2 and x_K and their asymptotes for the common unit-variance G-Γ pdfs.

Figures 4 and 5 compare distortions $D(Q_N^U)$ and $D(Q_N^*)$ with the corresponding asymptotic formulas listed in Table 2. The Hui-Neuhoff asymptotic constant C_{HN} to which $N^2D(Q_N^U)/(\ln N)^{2/\alpha}$ converges equals 2/3 for the two-sided Rayleigh and Laplacian, 4/3 for the Gaussian, and 16/9 for the gamma pdf and that the Panter-Dite asymptotic constant C_{PD} to which $N^2D(Q_N^U)$ converges equals $\sqrt{3}\pi/2 = 2.7207$ for the Gaussian, 9/2 for the Laplacian, $\frac{3}{4}\Gamma^3\left(\frac{2}{3}\right) = 1.8622$ for the two-sided Rayleigh, and $4\sqrt{3}\Gamma^3\left(\frac{5}{6}\right)/\sqrt{\pi} = 5.6219$ for the gamma pdf. These C_{PD} agree well with those in [6].

First of all we observe that $N^2D(Q_N^*)$ in all the cases increases monotonically to the respective C_{PD} and settle down approximately within 2% and 1% at R=7 and 8, respectively. On the other hand, the approach of $N^2 D(Q_N^U)/(\ln N)^{2/\alpha}$ to C_{HN} appears slow or even stagnant in some cases: at $R=16$, C_{HN} is approximately 20% above from the true value for the Gaussian pdf; 8% for the two-sided Rayleigh; 9% for the Laplacian; 32% for the gamma.

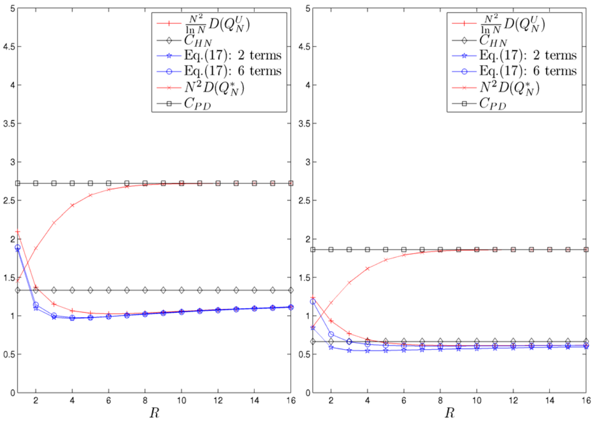

Fig. 4. Distortions $D(Q_N^U)$ and $D(Q_N^*)$ and their asymptotes for the unit-variance Gaussian (left) and two-sided Rayleigh (right) pdf.

The two-term formula for $D(Q_N^U)$, i.e., $C_{HN}(1+2c_1 \ln\ln N/(\alpha \ln N))$ obtained dropping the last term in Table 2, or the multi-term formula (17) with the dropped o term shows improved accuracy at the studied rates and the improvement is quite dramatic in many cases. For the Gaussian pdf, the two-term asymptote has a dramatically improved accuracy, resulting in an error of less than 1% for $R \geq 9$, and the six-term formula from Eq. (17) has essentially the same accuracy. (We note that, in the case of the Gaussian pdf, Eq. (17) has only five effective terms due to a zero coefficient for the $(\ln\ln N)^2/(\ln N)^2$ term.) For the two-sided Rayleigh pdf, the two-term formula underpredicts approximately 8% at $R=8$ to 4% at $R=16$, whereas the six-term formula from Eq. (17) (effectively only three terms due to zero coefficients) predicts with error approximately 1% or less for R=8 to 16, showing quite considerable improvement. For the Laplacian pdf, Eq. (17) has prediction error of approximately 2% for $R=9\sim16$ and its accuracy improves with R, while the two-term asymptote has poorer accuracy. For the gamma pdf, Eq. (17) has around 10% error at $R=9$ and its accuracy improves steadily, resulting in only 2% error at $R=16$.

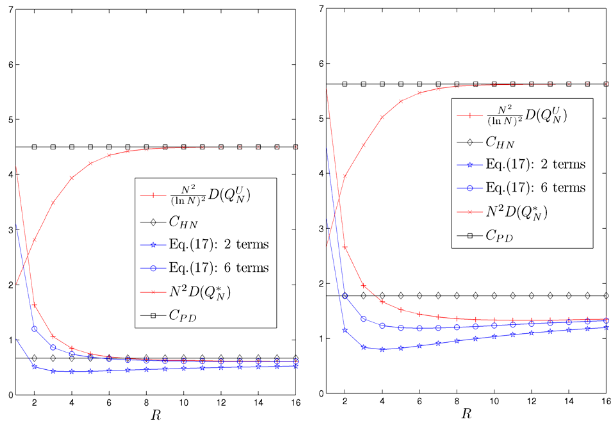

Fig. 5. Distortions $D(Q_N^U)$ and $D(Q_N^*)$ and their asymptotes for the unit-variance Laplacian (left) and gamma (right) pdf.

It is summarized that the distortion of uniform quantizers converges slowly while staying at a close proximity to the limit for the Rayleigh and Laplacian pdfs and at a distance to the limit for the Gaussian and gamma. Additional terms in the asymptote improve accuracy considerably and in many cases dramatically. Roughly speaking, when $R \geq 8$, the one-term asymptotic formula $D(Q_N^*) \approx C_{PD}/N^2$ for nonuniform quantizers and the multi-term formulas for uniform quantizers are considered accurate. Also from the observation we find the bounds for the common G-Γ pdfs: $R \geq 7$

$$\frac{C_{HN}(\ln N)^{2/\alpha}}{N^2}\left(1+\frac{2c_1}{\alpha}\frac{\ln\ln N}{\ln N}\right) \leq D(Q_N^U) \leq \frac{C_{HN}(\ln N)^{2/\alpha}}{N^2}$$

where $\frac{2c_1}{\alpha} = -\frac{2\alpha-\beta-1}{\alpha^2} < 0.$

V. Concluding Remarks

This paper compared the MSE distortion and the innermost step size x_2 and support limit x_K, as key characteristics, of optimal symmetric uniform and nonuniform quantizers with their corresponding asymptotic formulas for the generalized gamma density functions. The formulas for the innermost step size and support limit are considered generally accurate and the distortion of nonuniform quantizers also can be accurately predicted from the respective asymptotic formulas approximately at $R=8$ or greater, whereas the distortion of uniform quantizers may exhibit reluctant or stagnant local behavior of convergence at the studied rates up to 16, though it possibly stays close to the asymptotic constant. Additional terms in the asymptote exhibit quite considerably improved accuracy for all the studied common generalized gamma pdfs and, in many cases the improvement is dramatic. Roughly speaking, when $R \geq 8$, the asymptotic formulas (multi-term formulas especially in the case of uniform quantizers) can predict the characteristics of optimal quantizers quite accurately, whether it is the innermost step size, support limit or distortion.

References

[1] E. W. Stacy, "A generalization of the gamma distribution," *Ann. Math. Stat.*, vol. 33, no. 3, pp. 1187~1192, Sep. 1962.

[2] D. Hui and D. L. Neuhoff, "Asymptotic analysis of optimal fixed-rate uniform scalar quantization," *IEEE Trans. Inform*. Thy., vol. IT-47, pp. 957~977, Mar. 2001.

[3] J. Max, "Quantizing for minimum distortion," *IRE Trans. Inform*. Thy., IT-6, pp. 7~12, Mar. 1960.

[4] M. D. Paez and T. H. Glisson, "Minimum mean-squared-error quantization in speech PCM and DPCM systems," *IEEE Trans. Commun.*, COM-20, pp. 225~230, Apr. 1972.

[5] W. A. Pearlman and G. H. Senge, "Optimal quantization of the Rayleigh probability distribution," *IEEE. Trans. Commun.*, COM-27, pp. 101~112, Jan. 1979.

[6] N. S. Jayant and P. *Noll, Digital Coding of Waveforms: Principles and Applications to Speech and Video*, Prentice-Hall, 1984.

[7] P. Kabal, "Quantizers for the gamma distribution and other symmetrical distributions," *IEEE Trans. Acoust., Speech, Signal. Process.*, vol. ASSP-32, pp. 836~841, Aug. 1984.

[8] S. Na and D. L. Neuhoff, "Asymptotic MSE distortion of mismatched uniform scalar quantization," to appear in *IEEE Trans. Inform. Thy.*, vol. IT-58, May 2012.

[9] S. Na and D. L. Neuhoff, "On the support of MSE-optimal, fixed-rate, scalar quantizers," *IEEE Trans. Inform. Thy.*, vol. IT-47, pp. 2972~2982, Nov. 2001.

[10] P. F. Panter and W. Dite, "Quantization distortion in pulse-count modulation with nonuniform spacing of levels," *Proc. IRE*, vol. 39, pp. 44~48, Jan. 1951.

[11] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, 1974.

Table 2. Asymptotic formulas for optimal quantizer for the G-Γ pdfs

x_2^U	$\frac{2}{N}\left(\frac{2\ln N}{\lambda}\right)^{1/\alpha}\left\{1+\frac{c_1}{\alpha}\frac{\ln\ln N}{\ln N}+\frac{c_2}{\alpha}\frac{1}{\ln N}+\frac{(1-\alpha)c_1^2}{2\alpha^2}\frac{(\ln\ln N)^2}{(\ln N)^2}+\left(\frac{c_1^2}{\alpha}+\frac{(1-\alpha)c_1c_2}{\alpha^2}\right)\frac{\ln\ln N}{(\ln N)^2}\right\}$
x_K^U	$\left(\frac{2\ln N}{\lambda}\right)^{1/\alpha}\left\{1+\frac{c_1}{\alpha}\frac{\ln\ln N}{\ln N}+\frac{c_2}{\alpha}\frac{1}{\ln N}+\frac{(1-\alpha)c_1^2}{2\alpha^2}\frac{(\ln\ln N)^2}{(\ln N)^2}+\left(\frac{c_1^2}{\alpha}+\frac{(1-\alpha)c_1c_2}{\alpha^2}\right)\frac{\ln\ln N}{(\ln N)^2}\right\}$
D_i	$\frac{1}{3}\left(\frac{2}{\lambda}\right)^{2/\alpha}\frac{(\ln N)^{2/\alpha}}{N^2}\left\{1+\frac{2c_1}{\alpha}\frac{\ln\ln N}{\ln N}+\frac{2c_2}{\alpha}\frac{1}{\ln N}\right\}$
D_o	$\frac{1}{3\alpha}\left(\frac{2}{\lambda}\right)^{2/\alpha}\frac{(\ln N)^{(2-\alpha)/\alpha}}{N^2}$
$D(Q_N^U)$	$\frac{C_{HN}(\ln N)^{2/\alpha}}{N^2}\left\{1+\frac{2c_1}{\alpha}\frac{\ln\ln N}{\ln N}+\left(\frac{2c_2}{\alpha}+\frac{1}{\alpha}\right)\frac{1}{\ln N}\right\};\ C_{HN}=\frac{1}{3}\left(\frac{2}{\lambda}\right)^{2/\alpha}$
x_2^*	$\left(\frac{2}{N}\right)^{3/(\beta+3)}\left(\frac{3}{\lambda}\right)^{1/\alpha}\left\{\frac{\beta+2}{2\alpha}\Gamma\left(\frac{\beta+3}{3\alpha}\right)\right\}^{3/(\beta+3)}$
x_K^*	$\left(\frac{3\ln N}{\lambda}\right)^{1/\alpha}\left\{1-\frac{3\alpha-\beta-3}{3\alpha^2}\frac{\ln\ln N}{\ln N}+\frac{\ln\left(3\Gamma\left(\frac{\beta+3}{3\alpha}\right)\right)}{\alpha}\frac{1}{\ln N}\right\}$
D_i	$\frac{3^{(\beta-\alpha+3)/\alpha}}{\alpha^2\lambda^{2/\alpha}}\frac{\Gamma^3\left(\frac{\beta+3}{3\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\alpha}\right)}\frac{1}{N^2}$
D_o	$\frac{3^{(\beta+3)/\alpha}}{\alpha^2\lambda^{2/\alpha}}\frac{\Gamma^3\left(\frac{\beta+3}{\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\alpha}\right)}\frac{1}{N^3}$
$D(Q_N^*)$	$\frac{C_{PD}}{N^2};\ C_{PD}=\frac{3^{(\beta-\alpha+3)/\alpha}}{\alpha^2\lambda^{2/\alpha}}\frac{\Gamma^3\left(\frac{\beta+3}{3\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\alpha}\right)}$

Appendix: Tables of Numerical Data

Numerical data for the distortion and thresholds of optimal uniform symmetric quantizers Q_N^U and nonuniform symmetric quantizers Q_N^* are tabulated for the unit-variance pdfs.

Table 3. The Gaussian pdf

R	x_2^U	x_K^U	$D(Q_N^U)$	R	x_2^*	x_K^*	$D(Q_N^*)$
1		0.000e+000	3.634e-001	1		0.000e+000	3.634e-001
2	9.957e-001	9.957e-001	1.188e-001	2	9.816e-001	9.816e-001	1.175e-001
3	5.860e-001	1.758e+000	3.744e-002	3	5.005e-001	1.748e+000	3.455e-002
4	3.352e-001	2.346e+000	1.154e-002	4	2.582e-001	2.401e+000	9.501e-003
5	1.881e-001	2.822e+000	3.495e-003	5	1.320e-001	2.976e+000	2.505e-003
6	1.041e-001	3.226e+000	1.040e-003	6	6.684e-002	3.492e+000	6.442e-004
7	5.687e-002	3.583e+000	3.043e-004	7	3.366e-002	3.962e+000	1.635e-004
8	3.076e-002	3.907e+000	8.769e-005	8	1.689e-002	4.395e+000	4.119e-005
9	1.650e-002	4.207e+000	2.492e-005	9	8.463e-003	4.797e+000	1.034e-005
10	8.785e-003	4.489e+000	6.997e-006	10	4.236e-003	5.174e+000	2.589e-006
11	4.650e-003	4.757e+000	1.944e-006	11	2.119e-003	5.529e+000	6.480e-007
12	2.448e-003	5.012e+000	5.355e-007	12	1.060e-003	5.866e+000	1.621e-007
13	1.284e-003	5.256e+000	1.464e-007	13	5.299e-004	6.187e+000	4.053e-008
14	6.704e-004	5.491e+000	3.974e-008	14	2.650e-004	6.494e+000	1.013e-008
15	3.490e-004	5.718e+000	1.073e-008	15	1.325e-004	6.790e+000	2.534e-009
16	1.812e-004	5.939e+000	2.881e-009	16	6.625e-005	7.084e+000	6.334e-010

Table 4. The two-sided Rayleigh pdf

R	x_2^U	x_K^U	$D(Q_N^U)$	R	x_2^*	x_K^*	$D(Q_N^*)$
1		0.000e+000	2.146e-001	1		0.000e+000	2.146e-001
2	8.946e-001	8.946e-001	8.091e-002	2	9.721e-001	9.721e-001	7.315e-002
3	4.955e-001	1.487e+000	2.499e-002	3	5.811e-001	1.504e+000	2.236e-002
4	2.747e-001	1.923e+000	7.464e-003	4	3.530e-001	1.951e+000	6.306e-003
5	1.510e-001	2.266e+000	2.197e-003	5	2.134e-001	2.346e+000	1.687e-003
6	8.228e-002	2.551e+000	6.382e-004	6	1.282e-001	2.702e+000	4.373e-004
7	4.444e-002	2.799e+000	1.832e-004	7	7.669e-002	3.026e+000	1.114e-004
8	2.381e-002	3.024e+000	5.193e-005	8	4.575e-002	3.325e+000	2.813e-005
9	1.267e-002	3.232e+000	1.456e-005	9	2.725e-002	3.604e+000	7.068e-006
10	6.705e-003	3.426e+000	4.042e-006	10	1.622e-002	3.865e+000	1.771e-006
11	3.529e-003	3.611e+000	1.112e-006	11	9.648e-003	4.111e+000	4.434e-007
12	1.850e-003	3.787e+000	3.038e-007	12	5.738e-003	4.345e+000	1.109e-007
13	9.658e-004	3.955e+000	8.243e-008	13	3.412e-003	4.568e+000	2.774e-008
14	5.027e-004	4.117e+000	2.223e-008	14	2.029e-003	4.781e+000	6.936e-009
15	2.609e-004	4.274e+000	5.967e-009	15	1.206e-003	4.985e+000	1.734e-009
16	1.351e-004	4.426e+000	1.594e-009	16	7.174e-004	5.184e+000	4.336e-010

Table 5. The Laplacian pdf

R	x_2^U	x_K^U	$D(Q_N^U)$	R	x_2^*	x_K^*	$D(Q_N^*)$
1		0.000e+000	5.000e-001	1		0.000e+000	5.000e-001
2	1.087e+000	1.087e+000	1.963e-001	2	1.127e+000	1.127e+000	1.762e-001
3	7.309e-001	2.193e+000	7.175e-002	3	5.332e-001	2.380e+000	5.448e-002
4	4.610e-001	3.227e+000	2.535e-002	4	2.644e-001	3.724e+000	1.537e-002
5	2.800e-001	4.200e+000	8.713e-003	5	1.322e-001	5.126e+000	4.102e-003
6	1.657e-001	5.136e+000	2.913e-003	6	6.618e-002	6.560e+000	1.061e-003
7	9.610e-002	6.054e+000	9.486e-004	7	3.311e-002	8.013e+000	2.699e-004
8	5.484e-002	6.965e+000	3.014e-004	8	1.656e-002	9.474e+000	6.806e-005
9	3.088e-002	7.875e+000	9.373e-005	9	8.284e-003	1.094e+001	1.709e-005
10	1.720e-002	8.787e+000	2.860e-005	10	4.143e-003	1.241e+001	4.282e-006
11	9.484e-003	9.702e+000	8.587e-006	11	2.071e-003	1.388e+001	1.072e-006
12	5.189e-003	1.062e+001	2.542e-006	12	1.036e-003	1.535e+001	2.681e-007
13	2.819e-003	1.154e+001	7.433e-007	13	5.179e-004	1.682e+001	6.704e-008
14	1.522e-003	1.247e+001	2.151e-007	14	2.589e-004	1.829e+001	1.676e-008
15	8.179e-004	1.340e+001	6.163e-008	15	1.295e-004	1.976e+001	4.191e-009
16	4.375e-004	1.433e+001	1.752e-008	16	6.474e-005	2.123e+001	1.048e-009

Table 6. The gamma pdf

R	x_2^U	x_K^U	$D(Q_N^U)$	R	x_2^*	x_K^*	$D(Q_N^*)$
1		0.000e+000	6.667e-001	1		0.000e+000	6.667e-001
2	1.066e+000	1.066e+000	3.200e-001	2	1.268e+000	1.268e+000	2.318e-001
3	7.957e-001	2.387e+000	1.323e-001	3	5.274e-001	3.089e+000	7.047e-002
4	5.400e-001	3.780e+000	5.008e-002	4	2.299e-001	5.128e+000	1.961e-002
5	3.459e-001	5.189e+000	1.784e-002	5	1.008e-001	7.293e+000	5.185e-003
6	2.130e-001	6.603e+000	6.073e-003	6	4.411e-002	9.530e+000	1.334e-003
7	1.273e-001	8.021e+000	1.996e-003	7	1.927e-002	1.181e+001	3.382e-004
8	7.436e-002	9.444e+000	6.379e-004	8	8.404e-003	1.412e+001	8.517e-005
9	4.264e-002	1.087e+001	1.993e-004	9	3.663e-003	1.644e+001	2.137e-005
10	2.410e-002	1.231e+001	6.107e-005	10	1.595e-003	1.877e+001	5.352e-006
11	1.345e-002	1.376e+001	1.842e-005	11	6.946e-004	2.111e+001	1.339e-006
12	7.433e-003	1.521e+001	5.483e-006	12	3.024e-004	2.346e+001	3.349e-007
13	4.072e-003	1.668e+001	1.613e-006	13	1.316e-004	2.581e+001	8.375e-008
14	2.216e-003	1.815e+001	4.694e-007	14	5.731e-005	2.817e+001	2.094e-008
15	1.197e-003	1.962e+001	1.354e-007	15	2.494e-005	3.053e+001	5.237e-009
16	6.445e-004	2.112e+001	3.870e-008	16	1.086e-005	3.289e+001	1.309e-009

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